Boundaries and Unphysical Fixed Points in Dynamical Quantum Phase Transitions

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We show that dynamic quantum phase transitions (DQPT) in many situations involve renormalization group (RG) fixed points that are unphysical in the context of thermal phase transitions. In such cases, boundary conditions are shown to become relevant to the extent of even completely suppressing the bulk transitions. We establish these by performing an exact RG analysis of the quantum Ising model on scale-invariant lattices of different dimensions, and by analyzing the zeros of the Loschmidt amplitude. Further corroboration of boundaries affecting the bulk transition comes from the three-state quantum Potts chain, for which we also show that the DQPT corresponds to a pair of period-2 fixed points.

Dynamical quantum phase transitions (DQPT), a recent discovery of phase transitions, often periodic, in large quantum systems during time evolution [1–3], have generated a lot of interest because here time itself acts as the parameter inducing the transitions. Also, to be at a transition point, only time needs to be chosen properly without any requirement of fine-tuning of system parameters, unlike thermal transitions [4]. The signature of DQPT is the nonanalytic behaviour of various quantities in time around critical times t_c's. These transitions have now been shown in many models, like the transverse-field Ising model (TFIM), spin chains, quantum Potts models, the Kitaev model, and many others[1, 2, 6–10], and also observed experimentally[11, 12]. In spite of being a zero-temperature quantum phenomenon, DQPT is not determined by the quantum phase transitions of the system but rather seems related to the classical thermal criticalities of an associated system[10]. However, despite the use of many techniques so far, very few exact results are known on the scaling and universality in DQPT [10]. Moreover, the natures of the possible phases and the transitions remain to be properly classified, e. g., whether only equilibrium phases and transitions would suffice or there can be specialities of its own[10].

A general approach for phase transitions is the renormalization group (RG) framework [5] in terms of length-dependent effective parameters and their flows to the fixed points (FP), with the stable FPs determining the allowed phases, and the unstable ones (or separatrices) the phase transitions. In this Letter, we adopt an *exact* RG scheme for TFIM and the three-state quantum Potts chain (3QPC). Our exact results establish that there are DQPTs involving FPs that are *unphysical* in traditional thermal transitions. Second, we show that, for those unphysical FPs, boundary conditions (BC) are relevant and can even lead to a suppression of the transitions completely, unlike thermal cases where BCs do not affect the bulk transitions. Another surprising result is the emergence of a pair of period-2 FPs, never seen in the thermal context, that controls the DQPT in 3QPC, in contrast to the zero-temperature FP [2] for the Ising DQPT case. In short, our exact results bring out several distinctive features of DQPT, not to be found in equilibrium transitions.

If a quantum system, with Hamiltonian H, is prepared in a noneigenstate $|\psi_0\rangle$ and suddenly allowed to evolve, then the probability for the system to be in state $|\psi_0\rangle$ after time t is given by $P(t) = |L(t)|^2 \sim e^{-N\lambda(t)}$, where

$$L(t) = \langle\psi_0|e^{-itH}|\psi_0\rangle \sim e^{-Nf(t)}, \quad (\hbar = 1), \qquad (1)$$

is the Loschmidt amplitude with $f(t)$ and $\lambda(t) = 2\mathrm{Re}f(t)$ as the large-deviation rate functions[13] for a large system of $N(\to\infty)$ degrees of freedom. Often, $\lambda(t)$ and $f(t)$ show phase-transition-like nonanalyticities at time $t = t_c$. These phase transitions in time are the DQPTs[8, 10].

TFIM is defined on a lattice as $H_{\mathrm{I}} = H + H_\Gamma$, where

$$H = -J\sum_{\langle jk\rangle}\sigma_j^{\mathrm{z}}\,\sigma_k^{\mathrm{z}}, \quad H_\Gamma = -\Gamma\sum_j \sigma_j^{\mathrm{x}}, \quad (J, \Gamma > 0), \qquad (2)$$

σ_j^α being the Pauli matrices (α = x, y, z) at lattice site j, and $\langle jk\rangle$ denoting nearest neighbours[14]. The interaction favours an aligned state in the z direction [15], and H_Γ is the transverse field term that aligns the spins in the x direction. We may add a boundary term given by $H_{\mathrm{B}} = -h(\sigma_1^{\mathrm{z}} + \sigma_N^{\mathrm{z}})$, where the boundary field h acts only on the first and the Nth spins. Two special cases are $h = 0$ and $h \to \infty$ corresponding to open BC and fixed BC (both up in the z-direction) respectively. For periodic BC in one dimension, $H_B = -J\sigma_1^{\mathrm{z}}\,\sigma_N^{\mathrm{z}}$.

The TFIM is prepared in a product state $|\psi_0\rangle$ [16] with all spins aligned in the x direction, e.g., by $\Gamma \to \infty$. At time $t = 0$, we set $\Gamma = 0$. So, the magnet evolves with H of Eq. (2) and any boundary term mentioned above. This is the particular sudden quench we use in this Letter. By expressing $|\psi_0\rangle$ in terms of the eigenstates of H, the Loschmidt amplitude and the rate function per bond can be expressed as [10, 15],

$$L(y) = 2^{-N}\sum_C y^{-E_C/2J}, f(y) = -N_B^{-1}\,\ln L(y), \qquad (3)$$

respectively, where $y = e^{2zJ}$, N_B is the number of bonds, and, for generality, z is taken as a complex number. $L(y)$

is an analytic continuation of the partition function of the traditional nearest neighbour Ising model[17] defined for $1 \leq y < \infty$ on the real positive axis ($z = \beta$ being the inverse temperature). The quantum time evolution in Eq. (1) is given by the unit circle $|y| = 1$ ($y = e^{i2Jt}$) in the complex y plane. A phase transition—defined as the point of nonanalyticity of f—is expected along the unit circle if there are zeros or limit points of zeros of $L(y)$ on the path[17]. An isolated zero on the circle, in contrast, just indicates orthogonality of the evolved and the initial states. The S^1 (circle) topology guarantees (via winding numbers) that, if there are zeros on the circle, there will be periodic transitions in time.

In one-dimensional TFIM and 3QPC, similar DQPT occurs, viz., linear kinks in $f(t)$, despite the absence of any thermal transitions[1, 7]. For the two-dimensional TFIM, DQPT was found to be the same as the two dimensional Ising critical point[2]. However, the generality of these results has not yet been established. In this context, we focus on a class of exactly solvable models that would help us in alienating the specialities of DQPT.

We choose scale-invariant lattices for which the real space renormalization group (RSRG) can be implemented exactly. The lattices are constructed hierarchically by replacing a bond iteratively by a diamondlike motif of b branches[18, 20] as shown in Fig. 1. Such lattices appear naturally in approximate RSRG for usual lattices. Three cases are considered here, *(i)* $b = 1$ corresponding to a one-dimensional lattice, *(ii)* $b = 2$ which is two dimensional but not a Bravais lattice, and *(iii)* $b = 3$ as a fractal-type lattice.

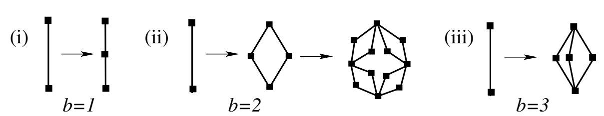

FIG. 1. Construction of hierarchical lattices. The sites are represented by squares. Replace each bond by a motif of b branches. *(i)* $b = 1$, *(ii)* $b = 2$ (diamondlike motif), and *(iii)* $b = 3$. Three generations are shown for $b = 2$.

The hierarchical structure of the lattice allows us to calculate $L(y)$ via a real space renormalization group approach, by decimating spins on individual motifs[18, 19]. Let us define $Z_n = 2^N L_n$ and $f_n = (2b)^{-n} \ln Z_n$ for the nth generation. Note that $f_n(y)$ is related to $f(y)$ of Eq. (3) by $f = f_n(1) - f_n(y)$. Z_n, and f_n satisfy the following recursion relations (see Supplemental Material [21])

$$Z_n(y) = \zeta(y_1) Z_{n-1}(y_1), \quad \zeta(x) = 2^b x^{1/2}, \tag{4a}$$

$$f_n(y) = (2b)^{-1} f_{n-1}(y_1) + (2b)^{-1} g(y_1), \tag{4b}$$

with $g(x) = \ln \zeta(x)$, and the RG flow equation

$$y_1 = 2^{-b}(y + y^{-1})^b. \tag{4c}$$

The boundary conditions (BC) are encoded in Z_1 as,

$$Z_1 = \begin{cases} 2(y^{1/2} + y^{-1/2}), & \text{(Open BC)} \\ y^{1/2}, & \text{(Fixed BC)}(\uparrow\uparrow), \end{cases} \tag{4d}$$

with $f_1 = \ln Z_1$.

Equation (4c) has FPs at $y = 1$ (infinite-temperature FP, paramagnetic phase), $y = \infty$ (zero-temperature FP, ordered phase), and a b-dependent unstable FP at $y = y_c$ (for $b > 1$) representing the critical point. For any odd $b > 1$, there are additional "unphysical" FPs at $y = -1, -y_c$ ($\pm\infty$ to be identified). There is no y_c for $b = 1$, as there is no thermal phase transition for the one-dimensional Ising model. The zeros of $L_n(y)$ can be determined from those of L_{n-1} via Eqs.(4a) and (4c), starting from the BC-dependent roots of $L_1(y) = 0$. In the $n \to \infty$ limit, the zeros then belong to the set of points that do not flow to infinity, thereby constituting the Julia set of the transformation[19]. These sets, obtained by MATHEMATICA, are shown for $b = 1, 2$, and 3 in Figs. 2 and 3.

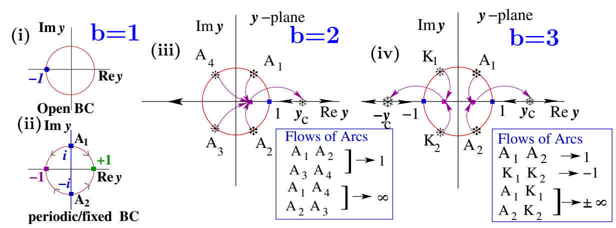

FIG. 2. Zeros of $L(y)$ in the complex-y plane, and RG flows. The red circle is the unit circle (UC) for time evolution. For $b = 1$, (i) only one zero at $y = -1$ for open BC, while (ii) the zeros populate the imaginary axis for periodic or fixed BC. (iii) For $b = 2$, the zeros meet the UC at four points, $\mathrm{A}_p, p = 1, 2, 3, 4$. Under RG, UC flows to the positive real axis, taking each A_p to $y_c = 3.38298....$ (iv) For $b = 3$, the four meeting points are of two types; A_1, A_2 flow to $y_c = 2.05817...$, while K_1, K_2 to $-y_c < 0$. See Fig. 3

The y, y^{-1} symmetry in Eq. (4c) ensures that if y^* is a FP, then $1/y^*$ flows to y^*. Therefore, there are four special points on the unit circle which flow to the nontrivial FPs, and are, necessarily, members of the Julia set. These four points on the unit circle in Figs. 2(iii), 2(iv), and Fig. 3 are the four critical points in time for $b > 1$. Incidentally, Eq. (4c) also ensures that any point on the unit circle, $y = e^{i\theta}$, under iteration, first flows to the real axis to $\cos\theta$ and then remains real afterwards. Consequently, complex RG fixed points for $b > 1$ are not important.

DQPT has been studied for $b = 1$ under periodic BC[1]. The surprising result we find here is that, unlike the thermal case, boundary conditions may even suppress the bulk transition. The transfer matrix solution of the 1D Ising model describes the partition function by the

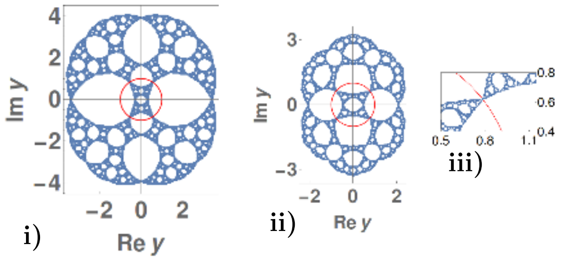

FIG. 3. Zeros of $L(y)$ as Julia sets in the complex-y plane for (i) $b = 2$, and (ii) $b = 3$. See Fig. 2. The zeros pinch UC at four points. (iii) For $b = 3$, zoomed view of the region near A_1 of Fig. 2(iv)

two eigenvalues $\Lambda_\pm = y^{1/2} \pm y^{-1/2}$, with the larger one determining the $N \to \infty$ behaviour[17]. For y flowing to $y^* = +1$ ($y^* = -1$), the larger eigenvalue in magnitude is Λ_+ (Λ_-), so that, with $y = e^{i\theta}$, the rate functions for the two regions ($f_\pm \sim \ln \Lambda_\pm/2$) are (see Supplemental material [21])

$$f_+(y) = -\frac{1}{2}\ln\cos^2\frac{\theta}{2}, \quad \text{and} \quad f_-(y) = -\frac{1}{2}\ln\sin^2\frac{\theta}{2}, \quad (5)$$

respectively. As characteristics of the high-temperature phases, $f_\pm$ should be independent of dimensions, remaining valid for all b. Open BC yields only one zero at $y = -1$ [Fig. 2(i)], and therefore no DQPT. On the other hand, periodic and fixed BC give zeros on the imaginary-y axis [Fig. 2(ii)]. Two zeros $y = \pm i$ on the unit circle, demarcating the RG flows of the points on the unit circle to $y = \pm 1$, are the known transition points[1, 2]. The transitions are from a paramagnetic (described by FP at $y = 1$, and f_+) to another paramagnetic phase, which we call para′, described by FP $y = -1$, and rate function f_-]. [Fig. 4(i)].

Now consider an open chain with the boundary term $H_\mathrm{B} = -h(\sigma_1^z + \sigma_N^z)$. For a finite chain, there will be contributions from both the FPs $y = \pm 1$, so that for an N-site chain (see Supplemental Meterial [21])

$$L(t, h) = (\cos Jt)^{N-1}\cos^2 ht + (i \sin Jt)^{N-1}\sin^2 ht. \quad (6)$$

DQPT with $f_\pm(t)$ is recovered in the $N \to \infty$ limit *only if* $h \neq 0$. See Fig. 4(i). For an open chain, $L(t) \equiv L(t, h = 0) = (\cos Jt)^{N-1}$. Hence, there is no transition [Fig. 4(ii)], consistent with one single zero [Fig. 2(i)]. There are four sectors of possible configurations of the two boundary spins, viz., $(\pm, \pm)$. Each of these four sectors individually shows DQPT. However, for the zero-field open chain, requiring superposition of the four sectors, there is a perfect cancellation of the $y = -1$ contributions. Thus, only f_+ survives [Fig. 4(ii)]. When the subtle cancellation of the four sectors is disturbed by the small boundary fields, the transitions appear, as shown in Fig. 4(i). We see that boundary conditions (like open-chain) become relevant only at the unphysical fixed point.

For any odd $b > 1$, there are four critical points on the unit circle, Figs. 2, and 3. These are A_1, $\theta_A = 2J\tau_1 = \arccos y_c{}^{-1/b}$, and A_2, $2J\tau_2 = 2\pi - \theta_A$ on the right half-plane, flowing to $y_c > 0$, and K_1, $2J\kappa_1 = \pi - \theta_A = \arccos(-y_c)^{-1/b}$, and K_2, $2J\kappa_2 = \pi + \theta_A$, on the left half-plane, flowing to the unphysical FP at $-y_c < 0$, via $y = -1/y_c$. These transition points ($l\pi \pm \theta_A$, for any integer l) are determined exactly. In this particular case, the nature of the singularity happens to be the same for all, as for the thermal case [a diverging third derivative of f, Fig. 4(iii)]. The flows of the four arcs of the unit circle are shown in Fig. 2(vi). $\mathrm{K}_1\mathrm{K}_2$, being characterized by FP $y = -1$, is expected to be sensitive to any constraint on the boundary spins. For, say, fixed boundary spins, a sequence of phases occurs in time, para-ferro-para′-ferro-para, separated by the four critical points. The two para phases with FP $y = \pm 1$ have ferromagnetic phases in between. However, in the unbiased case, the algebraic sum of the contributions of the four boundary sectors may lead to cancellation as in the $b = 1$ case. A signature of the cancellation in the $\mathrm{K}_1\mathrm{K}_2$ region is the failure of Eq. (4b) for f as $y \to -1$ on renormalization. This stability problem is also seen in the one-dimensional case, Fig. 4(i) *vis-à-vis* Fig. 4(ii) [the recursion relation, Eq. (4b), fails for $\pi/2 < 2Jt < 3\pi/2$]. We, therefore, conjecture that for the open BC case (free boundary spins) there is no intermediate para′ phase, but instead the whole arc $\mathrm{A}_1\mathrm{K}_1\mathrm{K}_2\mathrm{A}_2$ represents the ferro phase—a major boundary effect on bulk DQPT.

For even b, there are again four points on the unit circle [Figs. 2(iii) and 3(i)], $\mathrm{A}_i, (i = 1, 4)$, which have identical angular relations as the four points for odd b, except that here all flow to y_c in two steps via $y = 1/y_c$. All points in arcs $\mathrm{A}_1\mathrm{A}_4$ and $\mathrm{A}_2\mathrm{A}_3$ flow to ∞ implying an ordered state, while the remaining two arcs, $\mathrm{A}_1\mathrm{A}_2$ and $\mathrm{A}_3\mathrm{A}_4$, flow to 1, the disordered phase. Therefore, there is an oscillation between ordered (broken-symmetry) phase and the standard disordered phase with critical points at four different times. The nonanalytic features at the four critical times are the same as for the temperature-driven critical point at y_c. In essence, DQPT here follows closely the thermal transition.

To show the generality of the boundary effect, let us consider the three state Potts chain of N sites (3QPC) involving 3×3 matrices [7]. The interaction term is

$$H = -J\sum_j (\Omega_j^\dagger \Omega_{j+1} + \mathrm{H.c.}), \quad (7)$$

where $\Omega = \mathrm{diag}(1, e^{i2\pi/3}, e^{i4\pi/3})$. Analogous to the transverse field of Eq. (2), the spin flipping term for Potts spin is $H_\Gamma = -\Gamma\sum_j T_j$, where the elements of the 3×3 matrix T are given by $T^{\alpha\beta} = 1 - \delta_{\alpha\beta}, (\alpha, \beta = 1, 2, 3)$. Γ can be used to prepare the chain in a product state of equal-amplitude superpositions of the three states of each spin.

$$y_1 = R(y) \equiv (y^2+2)/(2y+1), \tag{8}$$

whose fixed points are $y_p = 1$, $y_u = -2$ ("unphysical"), and $y = \pm\infty$. The DQPT involves the transition between the two stable phases described by y_p and y_u, (analogous to Fig. 2(iv)), with the critical times at the points of intersection of the unit circle and the line of zeros of $L(y) = 0$. These intersectioons are A_1, $y_{A1} = e^{i2\pi/3}$, and A_2, $y_{A2} = e^{i4\pi/3}$, which *flow into each other* under the RG transformation. In other words, A_1,i and A_2 are period-2 FPs of RG, i.e., the *fixed points* of $R^{(2)} = R(R(y))$. By linearizing $R^{(2)}$ around A_1 or A_2, the thermal eigenvalue is found to be 1, which leads to a kink in $\mathrm{Re}\, f$ at the transition points [7]. The emergence

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[3] The periodic nature, when it happens, is one of the dramatic aspects of the transition. It should be mentioned that DQPT is not necessarily periodic in every situation.
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[15] For a lattice of N sites, the complete spectrum, $H|\phi_C\rangle = E_C|\phi_C\rangle$, is given by $|\phi_C\rangle = \bigotimes_{j=1,N} |\bullet\rangle_j$, and $E_C = -J\sum_{\langle jk\rangle} s_j s_k$, where $|\bullet\rangle_j$ is either $|\uparrow\rangle$ or $|\downarrow\rangle$ eigenstate of σ_j^z with $s_j = \pm 1$ in configuration C (there are 2^N states).
[16] The initial state is $|\psi_0\rangle = \bigotimes_j |\rightarrow\rangle_j$, where $|\rightarrow\rangle$is an eigenstate of σ^x. For TFIM, the paramagnetic and the ferromagnetic states are defined with respect to the net orientation in the z-direction. The state with all spins in the x-direction, by this definition, is a para state.
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[22] The RG equation can be obtained by comparing M^2 with M where M is the transfer matrix of Ref. [7].
[23] More details on the Potts model for different b will be reported elsewhere.
[24] Yantao Wu, arXiv/1906.07945.

Supplemental Material on "Boundaries and Unphysical Fixed Points in Dynamical Quantum Phase Transitions"

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I. ON HIERARCHICAL LATICES

For a hierarchical lattice (diamond type) of b branches, the dimension[S1] is given by

$$d = \frac{\ln 2b}{\ln 2}, \tag{S1}$$

so that $d = 1,\ 2$ for $b = 1,\ 2$, while $d = \ln 6/\ln 2 = 2.58...$ for $b = 3$. For the nth generation, the number of bonds is $B_n = (2b)^n$ and the number of sites is

$$N_n = bB_{n-1} + N_{n-1} = 2 + b\frac{(2b)^n - 1}{2b-1}. \tag{S2}$$

In the $n \to \infty$ limit, $\frac{N_n}{B_n} \to \frac{b}{2b-1}$.

II. PARTITION FUNCTION

The partition function of the traditional nearest neighbour Ising model is given by[S2]

$$Z \;=\; \sum_{C\in 2^N \text{states}} e^{-\beta E_C}, \tag{S3}$$

with

$$E_C = -J\sum_{\langle jk\rangle} s_j s_k,$$

as the energy of configuration C. Here, $s_j = \pm 1$, $\beta = 1/k_B T$, T being the temperature and k_B the Boltzmann constant. The free energy is given by $-k_B T \ln Z$.

We take $y = e^{2\beta J}$ as the variable because $2J$ is the energy gap for a single bond.

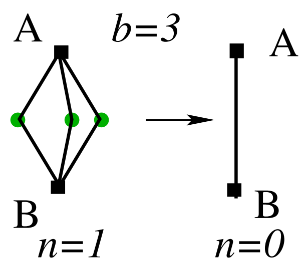

FIG. S1. Decimation of a motif of b branches by summing over the internal spins (green disks), keeping the spins at A and B fixed.

The existence of the thermodynamic limit ($N \to \infty$) for the Ising model ensures the large deviation form $L = \exp(-Nf(y))$, with f as the analog of the free energy extended to the complex plane.

RG RELATIONS FOR THE ISING MODEL

We derive the RG relations for the Ising model on a hierarchical lattice of b branches. The decimation of a motif to a bond is shown in Fig. S1

Denoting the partition function of a bond of two spins $s_1, s_2 = \pm 1$ by $Z_{s_1 s_2}(y)$, the summation over the internal spins (green disks) yields the relation between the partition function Z_1 of the larger cell (generation $n = 1$) as a function of y and the partition function of a bond ($n = 0$) as a function of the renormalized parameter y_1.

Explicitly,

$$Z_1|_{++} \equiv [(Z_{++}(y))Z_{++}(y) + Z_{+-}(y)Z_{-+}(y)]^b = \zeta(y_1)\ Z_{++}(y_1), \tag{S4a}$$

$$Z_1|_{+-} \equiv [Z_{+-}(y)Z_{--}(y) + Z_{++}(y)Z_{+-}(y)]^b = \zeta(y_1)\ Z_{+-}(y_1), \tag{S4b}$$

exploiting the fact that the b branches are independent. Only two relations are sufficient, thanks to the symmetry that $Z_{++}(y) = Z_{--}(y)$, and $Z_{+-}(y) = Z_{-+}(y)$. Noting that $Z_{++}(y) = y^{1/2}$, and $Z_{+-}(y) = y^{-1/2}$, one gets

$$y_1 = 2^{-b}(y + y^{-1})^b, \tag{S5a}$$

$$\zeta(y) = 2^b y^{1/2}. \tag{S5b}$$

III. DERIVATION OF $f_\pm$ AND ZEROS OF THE ONE-DIMENSIONAL ISING CHAIN

The two high temperature fixed points of Eq. S5a, $y^* = \pm 1$ characterize the two possible phases of the one dimensional Ising model. The Ising partition function can be determined by a transfer matrix approach [S2].

If T is the 2×2 transfer matrix, then the partition functions for a chain of N sites under different BCs are given by

$$Z_N = \begin{cases} \mathrm{Tr}\ \mathsf{T}^N, & \text{(periodic BC)},\\ \sum_{j,\,k=1,\,2}\ [\mathsf{T}^{N-1}]_{jk}, & \text{(free BC)},\\ (e^{zh}\ e^{-zh})\mathsf{T}^{N-1}\begin{pmatrix} e^{zh}\\ e^{-zh}\end{pmatrix}, & \text{(with boundary fields)}, \end{cases} \tag{S6}$$

For periodic BC

$$Z_N = \Lambda_+^N + \Lambda_-^N, \tag{S7}$$

where $\Lambda_\pm = y^{1/2} \pm y^{-1/2}$ are the two eigenvalues of T. Eq. 6 of the text follows from Eq. (S6)

For large N, and $y = e^{i\theta}$

$$Z_N = \begin{cases} \Lambda_+^N, & \text{for } \theta \text{ near } 0,\ |\Lambda_+| > |\Lambda_-|\\ \Lambda_-^N, & \text{for } \theta \text{ near } \pi,\ |\Lambda_-| > |\Lambda_+| \end{cases} \tag{S8}$$

The rate functions

$$f_\pm(\theta) \equiv -\mathrm{Re}\lim_{N\to\infty} \ln(Z_N/2^N) = -\mathrm{Re}\ln(\Lambda_\pm/2),$$

are then given by

$$f_+ = -\frac{1}{2}\ln\cos^2\frac{\theta}{2}, \quad \text{and} \quad f_- = -\frac{1}{2}\ln\sin^2\frac{\theta}{2}, \tag{S9}$$

as quoted in Eq. (5) of the text.

Zeros of the one-dimensional Ising chain

The partition function for an N-site Ising chain with periodic boundary condition is given in Eq. (S7). The zeros of $L_N(y) = Z_N/2^N$ are then given by

$$y = i\ \cot\frac{(2n+1)\pi}{N}, n = -N, ..., N-1,$$

which lie along the imaginary axis. This is shown in Fig 2(b) in the text.

For open BC, the partition function is given by $Z_N = \Lambda_+^N$. There is therefore only one zero at $\Lambda = 0$. Therefore, $L_N(y)$ has only one zero at $y = -1$, as shown in Fig 2(a) in the text.

V. ZEROS OF THE ONE-DIMENSIONAL POTTS CHAIN

For the Potts model $y = \exp(3\beta J)$. The partition functions for the Potts chain are given by

$$Z_N = \begin{cases} \Lambda_1^N + 2\Lambda_2^N, & \text{(periodic BC)},\\ \Lambda_1^N, & \text{(Open BC)}, \end{cases} \tag{S10}$$

where $\Lambda_1 \propto (y+2)$, and $\Lambda_2 \propto (y-1)$ are the two eigen values of the transfer matrix given in Ref. [S4]. The zeros for the periodic BC case are given by $\Lambda_1/\Lambda_2 = e^{i(2n+1)\pi/N}$, so that

$$y = -\frac{1}{2} + i\frac{3}{2}\cot\frac{\pi(2n+1)}{N}, \quad \text{(PBC)}. \tag{S11}$$

For an open chain (free BC), it follows from Eq. (S10) that there is only one zero at $\Lambda_1 = 0$, i.e., at $y = -2$.

V. ON L_n AND f_n

Under the decimation transformation of Fig. S1 and Eq. (S4a), the partition function for generation n with parameter y, $Z_n(y)$, can be related to that of the $(n-1)$th generation with parameter y_1. The recursion relations for Z_n, and $f_n = (2b)^{-n}\ln Z_n$ can be written as[S3]

$$Z_n(y) = \zeta(y_1)Z_{n-1}(y_1), \tag{S12}$$

$$\text{and } f_n(y) = \frac{1}{2b}f_{n-1}(y_1) + \frac{1}{2b}g(y_1), \tag{S13}$$

where $g(x) = \ln\zeta(x) = 2^{-1}\ln(4^b x)$, and y_1 is given by Eq. (S5a). Note that f_n is defined without the normalization factor 2^N (Eq. (4a) in text). With successive transformation $y \to y_1 \to ... \to y_n$ following the RG flow equation, the Loschmidt amplitude is given by a rapidly convergent sum for large n as

$$f_{n+1}(y) = \sum_{j=1}^{n}\frac{1}{(2b)^j}g(y_j) + \frac{1}{(2b)^n}f_1(y_n), \tag{S14}$$

provided the functions remain well-defined at the transformed arguments.

The fixed points of Eq.(S5a) satisfy the equation

$$y^2 - 2y^{(b+1)/b} + 1 = 0. \tag{S15}$$

The nontrivial fixed points are $y_c = 3.38298....$ for $b = 2$, and $y_c = 2.05817...$ for b = 3.

In the limit $n \to \infty$, one gets

$$f_\infty(1) = \frac{b}{2b-1} \ln 2, \tag{S16}$$

which is the infinite temperature entropy per bond Eq. (S2)). The rate function f as defined in Eq. (4a) in the text, is given by $f(y) \equiv f_n(1) - f_n(y)$, where $f_n(1)$ takes care of the normalization in L. Fig. 4 in the text shows the plots of $f(y)$.

For points A_i's and K_i's that flow to $\pm y_c$ in two steps, we have the same value of $f = f_c$ for all of them with

$$f_c = \frac{1}{4b} \ln \frac{4^b}{y_c} + \frac{1}{4b(2b-1)} \ln(4^b y_c) - \frac{b}{2b-1} \ln 2, \tag{S17}$$

which evaluates to

$$f_c\big|_{b=2} = 0.10156312, \quad \text{and} \quad f_c\big|_{b=3} = 0.0482183.$$

The intersection of the $f = f_c$ line with the f-vs-t curve gives the critical points A_i's and K_i's.

For $b = 3$, the transition points are

$$\begin{aligned} \mathrm{A}_1 &: 2J\tau_1 = 0.666239..., \\ \mathrm{A}_2 &: 2J\tau_2 = 2\pi - 0.666239 = 5.616946, \\ \mathrm{K}_1 &: 2J\kappa_1 = \pi - 0.666239 = 2.47535, \\ \text{and } \mathrm{K}_2 &: 2J\kappa_2 = 3.807832. \end{aligned}$$

For $b = 2$, the transition points are

$$\begin{aligned} \mathrm{A}_1 &: 2J\tau_1 = 0.99597..., \\ \mathrm{A}_2 &: 2J\tau_2 = 5.28722, \\ \mathrm{A}_3 &: 2J\tau_3 = 4.13756, \\ \text{and } \mathrm{A}_4 &: 2J\tau_4 = 2.14562. \end{aligned}$$

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