Eigenvalues and Diagonal Elements

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Abstract. A basic theorem in linear algebra says that if the eigenvalues and the diagonal entries of a Hermitian matrix are ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $a_1 \leq a_2 \leq \cdots \leq a_n$, respectively, then $\lambda_1 \leq a_1$. We show that for some special classes of Hermitian matrices this can be extended to inequalities of the form $\lambda_k \leq a_{2k-1}$, $k = 1, 2, ..., \lceil \frac{n}{2} \rceil$.

Key words: Hermitian matrix, Majorization, Nonnegative matrix, Laplacian matrix of graph.

Let A be an $n \times n$ complex Hermitian matrix. The eigenvalues and the diagonal entries of A are real numbers, and we enumerate them in increasing order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

respectively. Various inequalities relating these two n-tuples are known and are much used in matrix analysis. For example, we have

$$\lambda_1 \leq a_1 \quad \text{and} \quad \lambda_n \geq a_n. \tag{1}$$

These are subsumed in the majorization relations due to I. Schur: for $1 \leq k \leq n$

$$\sum_{j=1}^{k} \lambda_j \leq \sum_{j=1}^{k} a_j, \tag{2}$$

with equality when $k = n$. This is a complete characterization of two n-tuples that could be the eigenvalues and diagonal entries of a Hermitian matrix. In general, there are no further relations between individual λ_j and a_k. However, for large and interesting subsets of Hermitian matrices, it might be possible to find such extra relations. In [1] the authors consider eigenvalues of matrices associated with graphs. Let G be a simple weighted graph on n vertices and let A be the signless Laplacian matrix associated with G. Then, it is shown in [1] that $\lambda_2 \leq a_3$. This result is extended to other classes in [3]. One of these is the class $\mathcal{P}$ of Hermitian matrices whose off-diagonal entries are nonnegative. (In particular, this includes symmetric entrywise nonnegative matrices.) It is shown in [3] that if $A \in \mathcal{P}$, then $\lambda_2 \leq a_3$.

In this note we consider, in addition the class $\mathcal{P}$, another class $\mathcal{I}$ consisting of Hermitian matrices all whose off-diagonal entries are purely imaginary. We show that the inequality $\lambda_2 \leq a_3$ is valid for $A \in \mathcal{I}$ as well. The proof we give works for both the classes $\mathcal{P}$ and $\mathcal{I}$. Then we show that much more is true for the class $\mathcal{I}$. We show that in this case the inequality $\lambda_{n-1} \geq a_{n-2}$ also holds. Further, for all $1 \leq k \leq \lceil \frac{n}{2} \rceil$ we have $\lambda_k \leq a_{2k-1}$. We construct examples to show that neither of these results is true for the class $\mathcal{P}$.

Theorem 1. Let A be an $n \times n$ Hermitian matrix whose off-diagonal entries are either all nonnegative real numbers or all purely imaginary numbers. Then

$$\lambda_2 \leq a_3. \tag{3}$$

In case the off-diagonal entries are all purely imaginary, we also have

$$\lambda_{n-1} \geq a_{n-2}. \tag{4}$$

For the second class of matrices in Theorem 1, we can go further:

Theorem 2. Let A be an $n \times n$ Hermitian matrix whose off-diagonal entries are all purely imaginary. Then, for $1 \leq k \leq \lceil \frac{n}{2} \rceil$,

$$\lambda_k \leq a_{2k-1} \quad \text{and} \quad \lambda_{n-k+1} \geq a_{n-2k+2}. \tag{5}$$

We remark that in both (1) and (5) the second inequality follows from the first by considering $-A$ in place of A. Similarly (4) follows from (3). The argument cannot be used for the class $\mathcal{P}$.

Our proofs rely upon two basic theorems of matrix analysis. Let $\lambda_j(A)$, $1 \leq j \leq n$, denote the eigenvalues of a Hermitian matrix enumerated in the increasing order. Weyl's inequality says that if A and B are two $n \times n$ Hermitian matrices, then

$$\lambda_j(A+B) \leq \lambda_j(A) + \lambda_n(B), \qquad 1 \leq j \leq n. \tag{6}$$

Cauchy's interlacing principle says that if A_r is an $r \times r$ principal submatrix of A, then

$$\lambda_j(A) \leq \lambda_j(A_r), \qquad 1 \leq j \leq r. \tag{7}$$

See Chapter III of [2] for this and other facts used here.

Proof of Theorem 1. If P is a permutation matrix, then the increasingly ordered eigenvalues and diagonal entries of PAP^T are the same as those of A. So, for simplicity, we may assume that the diagonal entries of A are in increasing order. Let

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{12} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix}$$

be the top-left 3×3 submatrix of A. (Note $a_{jj} = a_j$ is our notation.) Decompose

$$A_3 = D_3 + M_3 \tag{8}$$

where D_3 is the diagonal part and M_3 the off-diagonal part of A_3. By Weyl's inequality

$$\lambda_2(A_3) \le \lambda_2(M_3) + \lambda_3(D_3) = \lambda_2(M_3) + a_3. \tag{9}$$

Note that $\det M_3 = 2\,\mathrm{Re}\, a_{12}a_{23}\overline{a_{13}}$. So, under the hypothesis of Theorem 1, $\det M_3 \ge 0$. We also have $\mathrm{tr} M_3 = 0$. These two conditions imply that we must have $\lambda_2(M_3) \le 0$. For, if $\lambda_3(M_3) \ge \lambda_2(M_3) > 0$, then the condition $\mathrm{tr} M_3 = 0$ forces $\lambda_1(M_3)$ to be negative. But this is impossible if $\det M_3 \ge 0$. So, from (9) we see that $\lambda_2(A_3) \le a_3$. Then, by the interlacing principle (7), we have $\lambda_2(A) \le a_3$. ■

Here we should observe that the only property of M_3 we used was that $\det M_3 \ge 0$. Thus the conclusion of Theorem 1 is valid for some other matrices not included in the classes $\mathcal{P}$ or $\mathcal{I}$.

Proof of Theorem 2. Let A_r be the top $r \times r$ principal submatrix of A. Decompose A_r as

$$A_r = D_r + M_r$$

where D_r is diagonal and M_r off-diagonal. The matrix iM_r is a real skew-symmetric matrix. So, the nonzero eigenvalues of iM_r are purely imaginary and occur in conjugate pairs. Thus the nonzero eigenvalues of M_r occur in $\pm$ pairs. This shows that

$$\lambda_k(M_r) \le 0 \quad \text{for} \quad 1 \le k \le \lceil \frac{r}{2} \rceil. \tag{10}$$

Now let $1 \le k \le \lceil \frac{n}{2} \rceil$. Using, successively, the interlacing principle, Weyl's inequality and (10), we get

$$\lambda_k(A) \le \lambda_k(A_{2k-1}) \le \lambda_k(M_{2k-1}) + a_{2k-1} \le a_{2k-1}.$$

■

We now give two examples to show why for the case of matrices with nonnegative off-diagonal entries we have to be content just with inequality (3). Let A be the 4×4 matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The 4×4 matrix E all whose entries are equal to one has eigenvalues $(4, 0, 0, 0)$. So the matrix $A = E - I$ has eigenvalues $(3, -1, -1, -1)$. Thus $\lambda_3 = -1$, and the inequality (4) does not hold in this case.

Let B be the 5×5 matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $B = S^2 + S^3$, where S is the shift matrix

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of S are the fifth roots of 1. Using this one readily sees that the eigenvalues of B are 2, $2\cos\frac{2\pi}{5}$ and $2\cos\frac{4\pi}{5}$, the first of these with multiplicity one and the latter two with multiplicities two each. In particular, $\lambda_3 > 0$ and the assertion $\lambda_3 \leq a_5$ in the first inequality (5) does not hold in this case.

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