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Localisation and Delocalisation for a Simple Quantum Wave Guide with Randomness

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Abstract. In this paper, we consider Schrödinger operators on $M \times \mathbb{Z}^{d_2}$, with $M = \{M_1, \ldots, M_2\}^{d_1}$ ('quantum wave guides') with a 'Γ-trimmed' random potential, namely a potential which vanishes outside a subset Γ which is periodic with respect to a sub-lattice. We prove that (under appropriate assumptions) for strong disorder these operators have *pure point spectrum* outside the set $\Sigma_0 = \sigma(H_{0,\Gamma^c})$ where H_{0,Γ^c} is the free (discrete) Laplacian on the complement Γ^c of Γ. We also prove that the operators have some *absolutely continuous spectrum* in an energy region $\mathcal{E} \subset \Sigma_0$. Consequently, there is a mobility edge for such models. We also consider the case $-M_1 = M_2 = \infty$, i.e. Γ-trimmed operators on $\mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$. Again, we prove localisation outside Σ_0 by showing exponential decay of the Green function $G_{E+i\eta}(x, y)$ uniformly in $\eta > 0$. For *all* energies $E \in \mathcal{E}$ we prove that the Green's function $G_{E+i\eta}$ is *not* (uniformly) in ℓ^1 as η approaches 0. This implies that neither the fractional moment method nor multi-scale analysis *can* be applied here.

1. Introduction

Quantum waveguides are quantum mechanical structures which are confined in certain spaces dimensions, but unconfined in others. The last decades showed a growing interest in these systems in the mathematical literature. The book [8] by Exner and Kovařík gives an overview on the state of art (as of 2015) as well as an extensive list of references on waveguides.

A quantum waveguide with the simplest geometry is given by a particle in a (k-dimensional) strip in $\mathbb{Z}^{k+m}$ or $\mathbb{R}^{k+m}$. Other examples are tubes or wires which are bended or twisted (see, for example, the discussion in Krejčiřík [21]). Of particular interest are waveguides with randomness either in the geometry of the system or in the potential energy.

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In this paper, we consider waveguides with a simple geometry, namely on a strip in $\mathbb{Z}^d$, for example, on

$$\mathcal{X} \;=\; \{M_1 p, M_1 p+1\ldots, M_2 p-1\}^{d_1}\times\mathbb{Z}^{d_2}, \qquad d_1+d_2=d, p\geq 2\,, \tag{1}$$

with a random potential V_ω. The potential we consider is 'sporadic' or 'Γ-trimmed', in the sense that $V_\omega(x)=0$ for lattice points $x\notin\Gamma$. Here, Γ is an $\mathbb{L}$-periodic subset of the strip for a sub lattice $\mathbb{L}$. In example (1), we may choose, for instance,

$$\Gamma=\{M_1 p,(M_1+1)p,\ldots,(M_2-1)p\}^{d_1}\times\mathbb{Z}^{d_2} \tag{2}$$

For $x\in\Gamma$, the potentials are independent and identically distributed. Random operators H_ω with such potentials are called 'Γ-trimmed'.

Spectral theory for trimmed Anderson models (i.e. on $\mathbb{Z}^d$) was done in the PhD-thesis of Obermeit [23] and the papers of Rojas-Molina [24], Elgart-Klein [5], Elgart-Sodin [6] and Kirsch-Krishna [16]. The present paper was inspired by [6].

We show that models as in (1), (2) have a mobility edge (or rather mobility edges). The measure theoretical nature of the spectrum depends on the energy region. More precisely, denote by H_{0,Γ^c} the free (discrete) Laplacian on the set Γ^c. Outside of the spectrum $\Sigma_0:=\sigma(H_{0,\Gamma^c})$ the operator H_ω has *dense point spectrum* for high enough disorder. On the other hand, we prove that H_ω has some *absolutely continuous spectrum* inside $\sigma(H_{0,\Gamma^c})$ regardless of the strength (or even existence) of the randomness.

The absolutely continuous spectrum comes from the existence of canonical extended states. More precisely, in an energy region inside Σ_0 we find periodic solutions of the free Schrödinger equation which vanish on the set Γ. Hence, these functions solve the Schrödinger equation with a random (in fact, with an arbitrary) potential on Γ as well.

Kotani and Simon consider random operators on a strip [20] with $d_2=1$. They give abstract conditions for absolutely continuous spectrum in terms of Lyapunov exponents. The examples for absolutely continuous spectrum they give are deterministic potentials. Their method does not extend to $d_2>1$.

To prove pure point spectrum we employ the multiscale analysis (see, for example, Dreifus-Klein [27] or Disertori et.al. [4] and references given there). The critical ingredient in our case is a *Wegner Estimate*. To prove this estimate, we use in an essential way that we work outside the spectrum $\sigma(H_{0,\Gamma^c})$. In fact, the estimate blows up when we approach $\sigma(H_{0,\Gamma^c})$.

From the multiscale bounds not only pure point spectrum follows but also dynamical localisation (see Damanik–Stollmann [3] or [26]).

We also consider the case $-M_1=M_2=\infty$, in other words a Γ-trimmed potential on $\mathbb{Z}^d=\mathbb{Z}^{d_1}\times\mathbb{Z}^{d_2}$. Again we use multiscale analysis outside Σ_0 for high enough disorder, which implies uniform exponential decay of the Green function and existence of pure point spectrum.

Inside a subset $\mathcal{E}$ of Σ_0 we prove that the Green function $G_{E+i\varepsilon}(x,y)$ is not only not uniformly exponentially decaying, but that for $x \notin \Gamma$ even

$$\sup_{\varepsilon \nearrow 0} \sum_{y \in \mathbb{Z}^d} |G_{E+i\varepsilon}(x,y)| \;=\; \infty\,. \tag{3}$$

Under appropriate conditions, we even have $\mathcal{E} = \Sigma_0$.

Consequently, we have a 'phase transition' at $\Sigma_0 = \sigma(H_{0,\Gamma^c})$ which manifests itself in the behaviour of the Green function.

For the Anderson model (with full randomness), it is expected that in higher dimension there is a mobility edge, namely a transition from pure point spectrum to absolutely continuous spectrum depending on the energy range and the strength of the disorder. All that is rigorously known (on $\mathbb{Z}^d$) is the existence of pure point spectrum (see, for example, Aizenman–Warzel [2] or Kirsch [15]). However, on the Bethe tree Klein [17] proved the existence of absolutely continuous spectrum (see also Klein–Sadel [19], Aizenman–Warzel [1], Froese et al. [9]).

There are random Schrödinger operators with decaying randomness for which a mobility edge is known to exist (see Krishna [22], Kirsch et.al. [10,13], and Jaksic-Last [11]. These models are *not* ergodic. However, the models we consider here are either ergodic in $\mathbb{Z}^{d_1}$-direction (for the strip) or even ergodic with respect to a d-dimensional sub-lattice.

The systems we consider are about the simplest wave guides possible. We expect that the localisation results we have in this paper can be extended to more complicated wave guide systems using essentially the same technique. We also expect the delocalisation results to hold more generally, assuming there are big enough regions without (random) potential. However, such results would presumably require refined methods.

2. Setup

We consider quantum systems (wave guides) on $\mathcal{X} \subset \mathbb{Z}^d = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$.

For $p = (p_1, \ldots, p_d) \in (\mathbb{N} \setminus \{1\})^d$ (periods) set (unit cell)

$$\mathcal{C}_0 \;= \{x \in \mathbb{Z}^d \mid 0 \le x_\nu \le p_\nu - 1 \text{ for all } \nu\} \tag{4}$$

By e_ν we denote the standard basis of $\mathbb{Z}^d$. The lattice $\mathbb{L}$ and the subset $\mathbb{L}_{M_1 M_2}$ are defined by

$$\mathbb{L} \;:=\; \Big\{ \sum_{\nu=1}^{d} i_\nu\, p_\nu e_\nu \mid i_\nu \in \mathbb{Z} \Big\} \tag{5}$$

$$\text{and} \qquad \mathbb{L}_{M_1 M_2} \;:=\; \Big\{ \sum_{\nu=1}^{d} i_\nu\, p_\nu e_\nu \mid M_1 \le i_\nu < M_2 \text{ for } \nu \le d_1 \Big\}, \tag{6}$$

with $M_1, M_2 \in \mathbb{Z}, M_1 < M_2$. In this paper, we always assume that $d_1, d_2 > 0$.

Then, we define 'cubes' $\mathcal{X}_{M_1 M_2}$ by 'periodising' $\mathcal{C}_0$:

$$\mathcal{X}_{M_1, M_2} \;:=\; \mathcal{C}_0 + \mathbb{L}_{M_1 M_2} \tag{7}$$

and

$$\mathcal{X}_\infty \;:=\; \mathcal{C}_0 + \mathbb{L}\,. \tag{8}$$

Informally, we consider $\mathcal{X}_\infty$ as $\mathcal{X}_{M_1M_2}$ with $M_1 = -\infty, M_2 = \infty$.

Thus, $\mathcal{X}_{M_1M_2}$ is a strip ('waveguide') of restricted width in d_1 directions and unconfined in d_2 directions, $\mathcal{X}_\infty$ is (for the moment) just a complicated expression for $\mathbb{Z}^d$.

Sometimes we omit the indices M_1, M_2 and ∞ if they are clear from the context or if they are irrelevant. For simplicity of arguments and, in fact, without loss of generality we assume that $M_2 - M_1$ is even.

The discrete Laplacian H_0 on $\mathcal{X}_\infty = \mathbb{Z}^d$ is given by:

$$H_0\, u(n) \;:=\; \sum_{\nu=1}^{d} \Big(u(n+e_\nu) \;+\; u(n-e_\nu) \Big) \tag{9}$$

When we restrict H_0 to subsets of $\mathbb{Z}^d$ we have to impose boundary conditions. In the following, we will most of the time work with either 'simple' boundary conditions or with 'periodic' boundary conditions.

Definition 2.1. If Λ is a subset of $\mathbb{Z}^d$, then the operator $H_{0,\Lambda}$ on $\ell^2(\Lambda)$ given by

$$H_{0,\Lambda} u(n) \;=\; \sum_{\nu=1}^{d} \Big(\chi_\Lambda(n+e_\nu)\, u(n+e_\nu) \;+\; \chi_\Lambda(n-e_\nu)\, u(n-e_\nu) \Big) \tag{10}$$

is called the Laplacian on Λ with *simple boundary conditions*.

Here

$$\chi_\Lambda(n) = \begin{cases} 1, \text{ for } \; n \in \Lambda; \\ 0, \text{ otherwise.} \end{cases} \tag{11}$$

Definition 2.2. Suppose the box $\Lambda \subset \mathbb{Z}^d$ is given by

$$\Lambda = \{ x \in \mathbb{Z}^d \mid q_\nu \le x_\nu \le p_\nu \text{ for } \nu = 1, \ldots, d' \} \tag{12}$$

for some $d' \le d$, then we call the operator H_0^Λ defined by

$$H_0^\Lambda u(x) \;=\; \sum_{\nu=1}^{d} \big(u(N_\nu^+ x) + u(N_\nu^- x) \big) \tag{13}$$

where

$$N_\nu^+ x \;=\; \begin{cases} x + e_\nu, & \text{if } \; x + e_\nu \in \Lambda; \\ x - (p_\nu - q_\nu) e_\nu, & \text{if } \; x + e_\nu \notin \Lambda. \end{cases} \tag{14}$$

$$N_\nu^- x \;=\; \begin{cases} x - e_\nu, & \text{if } x - e_\nu \in \Lambda; \\ x + (p_\nu - q_\nu) e_\nu, & \text{if } \; x - e_\nu \notin \Lambda. \end{cases} \tag{15}$$

the Laplacian on Λ with periodic boundary conditions.

For the operator H_0 on $\mathcal{X}_{M_1M_2}$, we impose *periodic* boundary conditions.

Now, we define the set Γ of 'active sites', i.e. the sites where the potential V_ω may be nonzero. The active sites *inside* $\mathcal{C}_0$ are denoted by Γ_0, with

$$\varnothing \neq \Gamma_0 \subsetneq \mathcal{C}_0 \tag{16}$$

and

$$\Gamma := \Gamma_0 + \mathbb{L}_{M_1 M_2} \tag{17}$$

where we include again the case $-M_1 = M_2 = \infty$.

Example 2.3. In the following examples $\mathcal{X}$ may be either $\mathcal{X}_{M_1M_2}$ or $\mathcal{X}_\infty = \mathbb{Z}^d$

1. For some $\nu \leq d_1$
$$\Gamma = \{x \in \mathcal{X} \mid x_\nu = 0\} \tag{18}$$
2. $\Gamma = \{x \in \mathcal{X} \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0\}$
3. For some results, we can deal with the following less restrictive model
$$\varnothing \neq \Gamma \subset \{x \in \mathcal{X} \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0\} \tag{19}$$

In this article, we investigate spectral properties of operators H on $\mathcal{X}$ of the form:

$$H\,u(n) = H_0\,u(n) + V(n)\,u(n) \tag{20}$$

where the potential V is supported by Γ, i.e. $V(n) = 0$ for $n \notin \Gamma$.

Most of the time we suppose that V is a random potential with independent, identically distributed random variables $V_\omega(\gamma), \gamma \in \Gamma$, but some of our results are independent of such an assumption.

3. Results

We prove localisation under fairly weak assumptions on Γ.

Theorem 3.1. *Suppose that $\varnothing \neq \Gamma \neq \mathcal{X}$. Assume that the random variables $V_\omega(n), n \in \Gamma$ are independent with a common distribution P_0 which has a bounded density ρ (with respect to Lebesgue measure) with compact support.*

If $I \subset \{E \mid \operatorname{dist}(E, \sigma(H_{0,\Gamma^c}) \geq \gamma\}$, then H_ω has pure point spectrum inside I with exponentially decaying eigenfunctions if $\|\rho\|_\infty$ is small enough, i.e. if $\|\rho\|_\infty \leq c_\gamma$.

Remark 3.2. Observe that $\|\rho\|_\infty$ small means high disorder.

The multiscale analysis gives a form of localisation which is more than merely pure point spectrum, namely dynamical localisation. Dynamical localisation appears in various forms (for a detailed discussion see, for example, [18]), we use it in the following form which we take from [26].

Definition 3.3. For a self-adjoint operator H and an interval I denote by $P_I(H)$ the spectral projection for H on I.

We say that the random operator H satisfies *dynamical localisation* in the energy interval I if for any $p > 0$

$$\mathbb{E}\left[\sup_{t>0} \left\| |x|^p|\, e^{-iHt}\, P_I(H)\varphi \right\|\right] < \infty \tag{21}$$

for any compactly supported φ.

Corollary 3.4. *Under the assumptions of Theorem 3.1, there is dynamical localisation in I.*

That dynamical localisation follows from multiscale analysis is proved in [3], see also [26].

The following result shows that Theorem 3.1 is not an empty statement.

Proposition 3.5. *Under the assumptions of Theorem 3.1, there is an $\eta > 0$ such that*

$$[\inf \Sigma, \inf \Sigma + \eta] \cap \sigma(H_{0,\Gamma^c}) \;=\; \varnothing \tag{22}$$

$$\textit{and} \qquad [\sup \Sigma - \eta, \sup \Sigma] \cap \sigma(H_{0,\Gamma^c}) \;=\; \varnothing \tag{23}$$

Theorem 3.1 and Corollary 3.4 are proved in Sect. 5. The proof of Proposition 3.5 is given in Sect. 4. These results reprove and extend previous results in [5,6,16,23,24].

Now, we turn to a class of examples for which we can prove the existence of *absolutely continuous spectrum.*

Definition 3.6. We call Γ as in (16) and (17) a *single layer set* if $\Gamma \subset G \times \mathbb{Z}^{d_2}$ with

$$\varnothing \;\neq\; G \;\subset\; G_0 = \{x \in \mathcal{X} \mid x_1 = 0 \text{ or } x_2 = 0 \text{ or } \ldots \text{ or } x_{d_1} = 0\} \tag{24}$$

We call Γ a *strict single layer set* if $\Gamma = G_0 \times \mathbb{Z}^{d_2}$.

Definition 3.7. We set

$$\mathcal{L} \;:=\; \bigtimes_{\nu=1}^{d_1} \{1, 2, \ldots, p_\nu - 1\} \tag{25}$$

$$\text{and for} \quad L \;:= (\ell_1, \ldots, \ell_{d_1}) \in \mathcal{L} \quad \text{we set}$$

$$e_L \;:= 2\Big(\sum_{\nu=1}^{d_1} cos\big(\frac{\pi \ell_\nu}{p_\nu}\big)\Big) \tag{26}$$

$$\mathcal{E} \;:=\; \{e_L \mid L \in \mathcal{L}\} \;+\; [-2d_2, 2d_2] \tag{27}$$

Proposition 3.8. *Consider the operator H_0 on $\mathcal{X}_{M_1,M_2}$, a set $\Gamma \subset \mathcal{X}_{M_1,M_2}$ and the operator H_{0,Γ^c}, the restriction of H_0 to Γ^c (with simple boundary conditions).*

1. *If Γ is a single layer set, then $\mathcal{E} \subset \Sigma_0 := \sigma(H_{0,\Gamma^c})$.*
2. *If Γ is a strict single layer set, then $\mathcal{E} = \Sigma_0$*

Theorem 3.9. *Consider H_0 on $\mathcal{X}_{M_1 M_2}$ with M_1, M_2 finite, $M_2 - M_1$ even and with periodic boundary conditions. Assume that Γ is a single layer set.*

If W is an arbitrary potential vanishing outside Γ, then

$$\mathcal{E} \;\subset\; \sigma_{ac}(H_0 + W)\,. \tag{28}$$

In particular, if Γ is a strict single layer set, then

$$\Sigma_0 \;\subset\; \sigma_{ac}(H_0 + W)\,. \tag{29}$$

Theorem 3.9 applies in particular to Γ-trimmed random potential as in Theorem 3.1. Thus, for such a random potential there is an energy region with pure point spectrum and a region with absolutely continuous spectrum. Consequently, there exists a *mobility edge*.

The proof of Theorem 3.9 is contained in Sect. 6.

Unfortunately, the proof of Theorem 3.9 does not work for the case $-M_1 = M_2 = \infty$, i.e. for $\mathcal{X} = \mathbb{Z}^d$. However, for this case we can at least show, that both the fractional moment method and the multiscale analysis cannot work. In fact, the spectral values in $\mathcal{E}$ belong to 'extended states' in an informal sense.

Let $\Gamma \subset \mathcal{X}_\infty$ be a one layer set and V_ω be a random potential on Γ satisfying the assumptions of Theorem 3.1. Denote by $G^{V_\omega}_{E+i\zeta}(x,y)$ the Green function (i.e. the kernel of the resolvent $(H_\omega - E - i\zeta)^{-1}$ with $\zeta > 0$).

Then, we show

Theorem 3.10. *We assume $\mathcal{X} = \mathbb{Z}^d$ and Γ is a single layer set.*

1. *If $E \notin \sigma(H_{0,\Gamma^c})$, then for high enough disorder*
$$\limsup_{\zeta \searrow 0} |G^{V_\omega}_{E+i\zeta}(x,y)| \;\le\; C\, e^{-m|x-y|} \tag{30}$$
$\mathbb{P}$-almost surely.
2. *If $E \in \mathcal{E}$, then*
$$\limsup_{\zeta \searrow 0} \sum_{y \in \mathbb{Z}^d} |G^{V_\omega}_{E+i\zeta}(x,y)| \;=\; \infty \tag{31}$$
for all $x \notin \Gamma$ and all ω.

Part 2 of Theorem 3.10 is actually a deterministic result, and it holds for *any* potential vanishing outside Γ.

We prove this theorem in Sect. 7.

4. The Random Operator and Its Spectrum

In this section, we consider operators with *random* potential. To emphasise this, we write
$$H_\omega \;:=\; H_0 + V_\omega\,. \tag{32}$$

Hypothesis 4.1. *We suppose that the potentials $V_\omega(\gamma), \gamma \in \Gamma$ are i.i.d. with a common distribution P_0. We assume that the support $\mathcal{S}$ of P_0 is compact.*

The assumption that $\mathcal{S}$ is compact can be relaxed considerably but we do not bother to do so.

We denote the corresponding probability space by $(\Omega, \mathcal{F}, \mathbb{P})$ which can and will be taken to be $\left(\mathcal{S}^\Gamma, \bigotimes_{\gamma\in\Gamma} \mathcal{B}(\mathcal{S}), \bigotimes_{\gamma\in\Gamma} P_0\right)$.

We write the lattice $\mathbb{L}$ as $\mathbb{L} = \mathbb{L}_1 \times \mathbb{L}_2$ with $\mathbb{L}_1 \subset \mathbb{Z}^{d_1}$ and $\mathbb{L}_2 \subset \mathbb{Z}^{d_2}$ and points x in $\mathcal{X}$ as $x = (x_1, x_2)$ with $x_i \in \mathbb{Z}^{d_i}$.

We define 'shift' operators $T_j, j \in \mathbb{L}' := \mathbb{L}_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$T_j\omega(x_1, x_2) \;:=\; \omega(x_1, x_2 - j) \tag{33}$$

It is easy to see that the shift T_j is measure preserving, i.e. $\mathbb{P}(T_j{}^{-1}A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$.

The following result tells us that the family $\{T_j\}_{j\in\mathbb{L}'}$ is *ergodic*:

Proposition 4.2. *If $A \in \mathcal{F}$ is invariant under $\{T_j\}_{j\in\mathbb{L}'}$, i.e. $T_j{}^{-1}A = A$ for all $j \in \mathbb{L}'$, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.*

This result can be found in [7], for example.

Define for $j \in \mathbb{L}'$ the shift operator

$$U_j u(x_1, x_2) := u(x_1, x_2 - j)\,, \tag{34}$$

for $(x_1, x_2) \in \mathcal{X}$.

The operators U_j are unitary on $\ell^2(\mathcal{X})$; moreover, the operators H_ω are *ergodic* in the sense

$$H_{T_j\omega} \;=\; U_j H_\omega U_j^*\,. \tag{35}$$

with ergodic T_j by Proposition 4.2

It follows (see, for example, [14]):

Proposition 4.3. *1. The spectrum $\sigma(H_\omega)$ is non-random (almost surely).*

2. The same is true for the measure theoretic parts of the spectrum (the absolutely continuous part $\sigma_{ac}(H_\omega)$, the singular continuous part, etc.).

3. There is (almost surely) no discrete spectrum.

Definition 4.4. We denote by Σ the almost sure spectrum of H_ω, i.e. $\Sigma = \sigma(H_\omega)$ $\mathbb{P}$-almost surely.

We now investigate the spectrum (as a set).

Definition 4.5. A function $W : \mathcal{X} \in \mathbb{R}$ is called an *admissible potential* (with respect to P_0) if

$$\begin{aligned} W(x) &\in \mathcal{S} = \operatorname{supp} P_0 && \text{if x} \in \Gamma, \\ W(x) &= 0 && \text{otherwise.} \end{aligned}$$

We denote the set of admissible potentials by $\mathcal{A}$.

Remark 4.6. Taking

$$(\Omega, \mathcal{F}, \mathbb{P}) \;=\; \Big((\mathcal{S})^\Gamma, \bigotimes_{\gamma\in\Gamma} \mathcal{B}(\mathcal{S}), \bigotimes_{\gamma\in\Gamma} P_0\Big) \tag{36}$$

there is a one-to-one correspondence τ between Ω and the set $\mathcal{A}$ of admissible potentials, namely $\tau(\omega)(n) = \sum_{\gamma\in\Gamma} \omega_\gamma \delta_{\gamma\, n}$.

We may therefore identify Ω and $\mathcal{A}$.

Theorem 4.7. *1. If W is an admissible potential, then $\sigma(H_0 + W) \subset \Sigma$.*

2. We have

$$\Sigma \;=\; \bigcup_{W\in\mathcal{A}} \sigma(H_0 + W)\,. \tag{37}$$

Proof. 1. For $E \in \sigma(H_0+W)$, there exists a Weyl sequence of functions φ_n with compact (hence finite) support; more precisely we may suppose that $\|\varphi_n\| = 1$ and

$$\|(H_0 + W - E)\,\varphi_n\| \ < \ \frac{1}{n} \tag{38}$$

Set $S_n = \operatorname{supp}\varphi_n$ which is a finite set. By the Borel–Cantelli lemma, there is a vector $j_n \in \mathbb{L}'$ such that

$$\sup_{k\in S_n} \ |\, W(k) - V_\omega(k + j_n)\,| \ < \ \frac{1}{n} \tag{39}$$

With $\psi_n(x) = \varphi_n(x + j_n)$, we therefore get

$$\|\,(H_\omega - E)\,\psi_n\,\| \ < \ \frac{2}{n}\,, \tag{40}$$

thus $E \in \sigma(H_\omega)$.

2. Since the set $\mathcal{A}$ has probability one (in the sense of Remark 4.6)

$$\Sigma \ \subset \ \bigcup_{W\in\mathcal{A}} \sigma(H_0 + W)\,. \tag{41}$$

This together with 1. proves the theorem. □

Set $V_a = a\chi_\Gamma$ then V_a is an admissible potential if $a \in \operatorname{supp} P_0$. We define $H_a = H_0 + V_a$ and set $E_{min}(a) = \inf\sigma(H_a)$ and $E_{max}(a) = \sup\sigma(H_a)$.

Theorem 4.8. *If* $\operatorname{supp} P_0 = [a, b]$*, then*

$$\Sigma \ = \ [E_{min}(a), E_{max}(b)]\,. \tag{42}$$

Proof. Since V_x are admissible for all $x \in [a, b]$ and due to continuity, we have $\Sigma \supset [E_{min}(a), E_{max}(b)]$.

Suppose now, that W is an admissible potential then by monotonicity

$$H_a \le H_0 + W \le H_b\,.$$

So, $\sigma(H_0 + W) \subset [\inf\sigma(H_a), \sup\sigma(H_b)]$. □

We will have a closer look at the operators H_a. Let us denote by $H_{0,\Gamma}$ and H_{0,Γ^c} the operator H_0 restricted to $\ell^2(\Gamma)$ and $\ell^2(\Gamma^c)$, respectively, with simple boundary conditions.

Proposition 4.9. *For any* $a \in \mathbb{R}$

$$\inf\sigma(H_a) \ \le \ \inf\sigma(H_{0,\Gamma^c}) \tag{43}$$

$$\textit{and} \ \sup\sigma(H_a) \ \ge \ \sup\sigma(H_{0,\Gamma^c})\,. \tag{44}$$

Moreover, if $a > 0$*, then*

$$\sup\sigma(H_a) \ > \ \sup\sigma(H_{0,\Gamma^c}) \tag{45}$$

Remark 4.10. For $a < 0$ we have $\inf\sigma(H_a) < \inf\sigma(H_{0,\Gamma^c})$.

Proof. Take $\varphi \in \ell^2(\Gamma^c)$ with $\|\varphi\|_{\ell^2(\Gamma^c)} = 1$ and define $\widetilde{\varphi} \in \ell^2(\mathcal{X})$ by

$$\widetilde{\varphi}(n) \;=\; \begin{cases} \varphi(n), \text{ for } \; n \in \Gamma^c; \\ \quad 0, \text{ otherwise.} \end{cases} \tag{46}$$

Then,

$$\langle \varphi, H_{0,\Gamma}\, \varphi \rangle_{\ell^2(\Gamma^c)} \;=\; \langle \widetilde{\varphi}, H_a \widetilde{\varphi}) \rangle_{\ell^2(\mathcal{X})} \,. \tag{47}$$

From (47), the equalities (43) and (44) follow by the Min-Max-Principle.

To prove the strict inequality (45), we show that $\sup H_a$ is strictly increasing with a.

The operator H_a is periodic; therefore, $E_a = \sup \sigma\,(H_a)$ is given by the $\sup h_a$ where h_a is the operator H_a restricted to the periodic cell C_0 with periodic boundary conditions. The corresponding eigenfunction ψ_a is strictly positive.

By the Hellmann–Feynman theorem

$$\frac{d}{da} E_a \;=\; \sum_{j \in C_0 \cap \Gamma} |\psi_a(j)|^2 \;>\; 0\,, \tag{48}$$

hence $E_a < E_b$ if $a < b$. This proves 45. The inequality in Remark 4.10 is proved in a similar way. □

Now, we consider the case that $\Gamma = G \times \mathbb{Z}^{d_2}$. In this case, the operator H_a separates in the following way.

Definition 4.11. The space $\mathcal{X}$ splits in a part $\mathcal{Z} \subset \mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$, namely

$$\mathcal{X}_\infty \;=\; \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \tag{49}$$

$$\mathcal{X}_{M_1 M_2} \;=\; \mathcal{Z} \times \mathbb{Z}^{d_2} \tag{50}$$

We denote the Laplacian on $\ell^2(\mathcal{Z})$ (possibly with periodic boundary conditions) by $H_0^{(1)}$ and the Laplacian on $\ell^2(\mathbb{Z}^{d_2})$ by $H_0^{(2)}$. We also set $H_a^{(1)} = H_0^{(1)} + a\chi_G$.

Then

$$H_a \;=\; \big(H_a^{(1)} \otimes \mathbf{1}_{\mathbb{Z}^{d^2}}\big) \;\oplus\; \big(\mathbf{1}_{\mathcal{Z}} \otimes H_0^{(2)}\big) \tag{51}$$

Consequently

$$\sigma(H_a) = \sigma(H_a^{(1)}) + [-2d_2, 2d_2]\,. \tag{52}$$

This proves the following Corollary:

Corollary 4.12. *If* $\Gamma = G \times \mathbb{Z}^{d_2}$ *and* $\operatorname{supp} P_0 = [a, b]$ *then*

$$\Sigma \;=\; [\inf \sigma(H_a^{(1)}), \sup \sigma(H_b^{(1)})] \;+\; [-2d_2, 2d_2] \tag{53}$$

The above consideration also allows us to prove Proposition 3.8.

Proof. (Proposition 3.8) Part 1. follows by computation. Part 2. follows from (52) □

5. Localisation

In this section, we prove pure point spectrum with exponentially decaying eigenfunction for energies $E \notin \sigma(H_{0,\Gamma^C})$ for high enough disorder.

For the ($\mathbb{L}$-periodic) set Γ of 'active' sites, we merely assume that $\Gamma \neq \varnothing$. We may also assume that $\Gamma \neq \mathcal{C}_0$, since otherwise Γ is the whole space, a case which is known, of course.

In the following, we consider the case $\mathcal{X} = \mathcal{X}_\infty$, at the end of this section we comment on the case $\mathcal{X}_{M_1 M_2}$.

We will use multiscale analysis (see [15] and references given there). During the proof we need to consider boxes Λ which are unions of shifted $\mathcal{C}_0$. Recall that the unit cell $\mathcal{C}_0$ is defined by:

$$\mathcal{C}_0 \;=\; \{x \in \mathbb{Z}^d \mid 0 \leq x_\nu \leq p_\nu - 1 \text{ for all } \nu\} \tag{54}$$

Definition 5.1. We call a set $\Lambda \subset \mathbb{Z}^d$ a $\mathcal{C}_0$-box, if

$$\Lambda \;=\; \{x \in \mathbb{Z}^d \mid L_\nu p_\nu \leq x_\nu \leq L'_\nu p_\nu - 1 \text{ for all } \nu\} \tag{55}$$

One of the crucial ingredients of (most versions of) multiscale analysis is the Wegner–Estimate. To prove this, we need the following result. By H_ω^Λ, we denote the operator $H_\omega = H_0 + V_\omega$ restricted to $\ell^2(\Lambda)$ with periodic boundary conditions.

Proposition 5.2. *Suppose Λ is a $\mathcal{C}_0$-cube and $E \notin \sigma(H_{0,\Gamma^c})$ and*

$$H_\omega^\Lambda \psi \;=\; E\psi \tag{56}$$

then

$$\|\psi\|_{\ell^2(\Lambda)} \;\leq\; \frac{C}{\operatorname{dist}\big(E, \sigma(H_{0,\Gamma^c})\big)}\, \|\psi\|_{\ell^2(\Lambda\cap\Gamma)} \tag{57}$$

with a constant C which is independent of Λ and E.

Proof. Set $\Lambda_1 = \Lambda \cap \Gamma^c$ and $\Lambda_2 = \Lambda \cap \Gamma$. We write $\ell^2(\Lambda)$

$$\ell^2(\Lambda) \;=\; \ell^2(\Lambda_1) \oplus \ell^2(\Lambda_2) \tag{58}$$

Accordingly, we may write the operator H_ω in block matrix form:

$$H_\omega^\Lambda \;=\; \begin{pmatrix} H_0^{\Lambda_1} & T \\ T^* & H_0^{\Lambda_2} + V_\omega \end{pmatrix} \tag{59}$$

The operator $T : \ell^2(\Lambda_2) \to \ell^2(\Lambda_1)$ 'restores' the links between Λ_2 and Λ_1.

The eigenvalue equation (56) reads

$$\begin{pmatrix} H_0^{\Lambda_1} & T \\ T^* & H_0^{\Lambda_2} + V_\omega \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \;=\; E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{60}$$

So, in particular

$$(H_0^{\Lambda_1} - E)\, \psi_1 \;=\; -T\, \psi_2 \tag{61}$$

Thus,

$$\|\psi\|_{\ell^2(\Lambda_1)} = \|\psi_1\| \tag{62}$$
$$\leq \|(H_0^{\Lambda_1} - E)^{-1}\| \, \|T\| \, \|\psi_2\| \tag{63}$$

Since for given m we have $|T(n,m)| = 1$ for at most $2d$ point n and $T(n,m) = 0$ otherwise, we conclude $\|T\| \leq 2d$.

Since we consider periodic boundary conditions on Λ, we have

$$\sigma(H_0^{\Lambda_1}) \subset \sigma(H_{0,\Gamma^c}) \tag{64}$$

as any eigenfunction on Λ can be periodically extended to an eigenfunction on $\mathcal{X}$. □

Now we turn to the Wegner estimate. By $N(A,E)$ we denote the number of eigenvalues of the operator A up to E.

Theorem 5.3. *If* $\operatorname{dist}\big(E, \sigma(H_{0,\Gamma^c})\big) \geq \gamma$ *and* $0 \leq \varepsilon \leq \frac{1}{2}\gamma$, *then*

$$\mathbb{E}\Big(N\big(H_\omega(\Lambda), E+\varepsilon\big) - N\big(H_\omega(\Lambda), E-\varepsilon\big)\Big) \leq \frac{C}{\gamma} \, \|\rho\|_\infty \, \varepsilon \, |\Lambda| \tag{65}$$

where $|\Lambda|$ *denote the volume (number of points) of* Λ.

Proof. The proof is a combination of the proofs from [15] and [12]. We sketch the main ideas. We use the abbreviation $H = H_\omega(\Lambda)$.

Let g be a monotone C^∞-function with $0 \leq g(t) \leq 1$, $g(t) = 0$ for $t \leq -2\varepsilon$ and $g(t) = 1$ for $t \geq 2\varepsilon$.

We obtain

$$N(H, E+\varepsilon) - N(H, E+\varepsilon) \leq \operatorname{tr} g(H - E + 2\varepsilon) - \operatorname{tr} g(H - E - 2\varepsilon) \tag{66}$$
$$= \int_{E-2\varepsilon}^{E+2\varepsilon} \operatorname{tr} g'(H - \lambda) \, \mathrm{d}\lambda \tag{67}$$

Let E_n denote the eigenvalues of $H_\omega(\Lambda)$ labelled in increasing order. These eigenvalues depend on the values $v_j := V_\omega(j), j \in \Lambda_2$.

Thus, we may consider

$$\sum_{j\in\Lambda_2} \frac{\partial}{\partial v_j} \operatorname{tr} g(H-\lambda) = \sum_n \sum_{j\in\Lambda_2} \frac{\partial}{\partial v_j} g(E_n - \lambda) \tag{68}$$
$$= \sum_n g'(E_n - \lambda) \sum_{j\in\Lambda_2} \frac{\partial E_n}{\partial v_j} \tag{69}$$
$$\geq \sum_n g'(E_n - \lambda) \sum_{j\in\Lambda_2} |\psi_n(j)|^2 \tag{70}$$
$$\geq C' \operatorname{dist}\big(E, \sigma(H_{0,\Gamma^c})\big) \operatorname{tr} g'(H-\lambda) \tag{71}$$

where ψ_n is a normalised eigenfunction of H with eigenvalue E_n. Above we used the Hellmann–Feynman theorem and, in the final step, Proposition 5.2.

Summarising we proved

$$\mathbb{E}\Big(N(H,E+\varepsilon)-N(H,E+\varepsilon)\Big) \tag{72}$$

$$\leq \frac{C''}{\operatorname{dist}\big(E,\sigma(H_{0\Gamma^c})\big)}\sum_{j\in\Lambda_2}\int_{E-2\varepsilon}^{E+2\varepsilon}\mathbb{E}\Big(\frac{\partial}{\partial v_j}\operatorname{tr} g(H-\lambda)\Big)\,\mathrm{d}\lambda \tag{73}$$

Suppose $\operatorname{supp} P_0 \subset [a,b]$ and denote by $H(v_j=c)$ the operator H with V_j replaced by the value c, then

$$\int \frac{\partial}{\partial v_j}\operatorname{tr} g(H-\lambda)\rho(v_j)\,dv_j \tag{74}$$

$$\leq \|\rho\|_\infty\,\Big(\operatorname{tr} g\big(H(v_j=b)-\lambda\big)-\operatorname{tr} g\big(H(V_j=a)-\lambda\big)\Big)\leq \|\,\rho\|_\infty \tag{75}$$

We used that changing the potential at one site j is a rank one perturbation and $0\leq g(\lambda)\leq 1$.

Performing the integrals over the $v_k, k\neq j$ gives the desired result. □

Once we have the Wegner estimate, the multiscale analysis follows the usual path. We need an initial length scale estimate and the induction step over growing length scales.

The initial length scale estimate follows directly from the Wegner estimate Theorem 5.3. In fact, as long as we are away from the spectrum of H_{0,Γ^c}, we can make the right hand side of (65) as small as we like by taking $\|\rho\|_\infty$ small. This corresponds to high disorder. Using a Combes—Thomas estimate, this allows us to prove the initial scale estimate. For details see Section 11.1 in [15].

The induction step follows the lines in [15] sections 9 and 10. The only difference being that we deal with periodic boundary conditions while [15] uses simple boundary conditions.

If $\mathcal{X}=\mathcal{X}_{M_1M_2}$, we start the induction with a cube of the form:

$$\Lambda=\{x\in\mathcal{X}_{M_1M_2}\mid L_\nu p_\nu\leq x_\nu\leq L'_\nu p_\nu-1 \text{ for } \nu=d_1+1,\ldots,d_1+d_2\} \tag{76}$$

Corollary 3.4 follows from the work [3] of Damanik and Stollmann.

6. Absolutely Continuous Spectrum

In this section we consider a special case of the above operators.

We start with the following observation:

Proposition 6.1. *Assume $\Gamma=G\times\mathbb{Z}^{d_2}$ and suppose the (otherwise arbitrary) potential W is concentrated on Γ. If there exists a polynomially bounded solution ψ of*

$$H_0^{(1)}\psi \;=\; e\,\psi \tag{77}$$

which vanishes on G, then

$$e\,+\,[-2d_2,2d_2]\;\subset\;\sigma(H_0+W) \tag{78}$$

Remark 6.2. $H_0^{(1)}$ and $H_0^{(2)}$ were defined in Definition 4.11.

Proof. Any $\eta \in [-2d_2, 2d_2]$ is of the form $\eta = 2\sum_{\nu=1}^{d_2} \cos(\kappa_\nu)$ and $\varphi(x) = \prod_{\nu=1}^{d_2} \sin(\kappa_\nu x_\nu)$ is a (bounded) function solving

$$H_0^{(2)}\varphi \;=\; \eta\,\varphi\,. \tag{79}$$

This can be verified by applying the addition theorem for the sinus.

Consequently, $\Psi(x,y) := \psi(x)\,\varphi(y)$ is a bounded solution to

$$H_0\,\Psi \;=\; (e+\eta)\,\Psi\,. \tag{80}$$

Since ψ vanishes on G, Ψ vanishes on Γ, so

$$(H_0+W)\,\Psi \;=\; H_0\,\Psi \;=\; (e+\eta)\,\Psi\,. \tag{81}$$

Thus, $e+\eta$ is a generalised eigenvalue of H_0+W. By Sch'nol's theorem any generalised eigenvalue belongs to the spectrum (see [25], Section C4 or [15], Section 7.1). □

We discuss a class of examples for which Proposition 6.1 applies.

We look at $\mathcal{X}$ or at the strip $\mathcal{X}_{M_1M_2}$. In the latter case, we impose periodic boundary conditions and take M_2-M_1 is even. This way we avoid to discuss various cases separately.

For $L=(\ell_1,\ldots,\ell_{d_1}), \ell_\nu \in \{1,2,\ldots,p_\nu-1\}$ we set

$$\Phi_L(x_1,\ldots,x_{d_1}) \;:=\; \prod_{\nu=1}^{d_1} \sin\Big(\frac{\pi\ell_\nu}{p_\nu}x_\nu\Big) \tag{82}$$

Lemma 6.3. *Under condition (24), the function $\Phi_L(x)$ is a solution to*

$$(H_0^{(1)}+W)\psi \;=\; 2\Big(\sum_{\nu=1}^{d_2} cos\big(\frac{\pi\ell_\nu}{p_\nu}x_\nu\big)\Big)\,\psi \tag{83}$$

and

$$\Phi_L(x) \;=\; 0 \qquad\qquad \textit{for}\;\; x\in G \tag{84}$$

Proof. Again by applying addition theorems and the fact that Φ_L vanishes on G_0, hence on G, we see that (83) holds. Moreover, since M_2-M_1 is even Φ_L satisfies periodic boundary conditions. □

Now, we are ready to prove Theorem 3.9.

Proof. (Theorem 3.9) Take $E\in\mathcal{E}$, then $E=e_L+\eta$ for some $\eta\in[-2d_2,2d_2]$. Denote by E_L the (one-dimensional) subspace of $\ell^2(\mathcal{Z})$ generated by the eigenfunction Φ_L.

The (closed) subspace $\mathfrak{h}_L = E_L\otimes\ell^2(\mathbb{Z}^{d_2})$ of $\ell^2(\mathcal{X})$ is invariant under the operator H_0+W and restricted to $\mathfrak{h}_L$ the operators H_0+W and H_0 agree. Consequently, H_0+W on $\mathfrak{h}_L$ is unitarily equivalent to $H_0^{(2)}+e_L$ on $\ell^2(\mathbb{Z}^{d_2})$, an operator with purely absolutely continuous spectrum.

7. Absence of Exponential Localisation

We start with a general observation. Let W be an arbitrary potential and denote by $G_z^W(x,y)$ the Green's function for $H := H_0 + W$, i.e. the kernel of the operator $(H_0 + W - z)^{-1}$.

Theorem 7.1. *If for some $E \in \mathbb{R}$*

$$H\,\psi \;=\; \big(H_0 + W\big)\,\psi \;=\; E\,\psi \tag{85}$$

for a bounded function ψ, then for all $x \in \mathbb{Z}^d$ with $\psi(x) \neq 0$

$$\liminf_{\zeta \searrow 0}\;\zeta \sum_{y \in \mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \;>\; 0\,, \tag{86}$$

in particular

$$\sup_{\zeta \searrow 0} \sum_{y \in \mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \;=\; \infty\,. \tag{87}$$

Observe that $|G_{E+i\zeta}(x,y)| \leq \frac{C}{\eta}\,e^{-c\eta\|x-y\|}$ by the Combes–Thomas estimate (see, for example, [15]). Thus, for any $\zeta > 0$

$$\sum_{y \in \mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \;<\; \infty\,. \tag{88}$$

Proof. Take $\varepsilon > 0$ arbitrary and assume that $|\psi(x)| \leq A < \infty$.

Let

$$\begin{aligned} \Lambda_L \;&=\; \{n \in \mathbb{Z}^d \mid |n_\nu| \leq L \text{ for } \nu = 1,\ldots,d\} \\ \text{and} \qquad \partial'\Lambda_L \;&=\; \{n \in \mathbb{Z}^d \mid |n_\nu - L| \leq 2 \text{ for some } \nu\} \end{aligned}$$

Denote by χ_L the characteristic function of Λ_L and set $\psi_L := \chi_L\,\psi$.

We compute

$$\begin{aligned} H\,\psi_L(x) \;&=\; \chi_L(x)\big(H\psi\big)(x) \;+\; \sum_{|j|=1} \psi(x+j)\big(\chi_L(x+j) - \chi_L(x)\big) \\ &=\; E\,\psi_L(x) \;+\; \sum_{|j|=1} \psi(x+j)\big(\chi_L(x+j) - \chi_L(x)\big)\,. \end{aligned}$$

It follows that for L big enough

$$\psi(x) = \sum_{y \in \mathbb{Z}^d} G^W_{E+i\zeta}(x,y)\,\Big(\sum_{|e|=1} \psi(y+e)\big(\chi_L(y+e) - \chi_L(y)\big) \;-\, i\zeta\psi(y)\chi_L(y)\Big)$$

Observe that $\chi_L(x+j) - \chi_L(x)$ (with $|j| = 1$) vanishes outside $\partial'\Lambda_L$. Consequently,

$$|\psi(x)| \;\leq\; 2d\,A \sum_{y \in \partial'\Lambda_L} |G^W_{E+i\zeta}(x,y)| \;+\; \zeta \sum_{y \in \mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \tag{89}$$

Since $\sum_{y\in\mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| < \infty$ for $\zeta > 0$, we can choose L (depending on $\zeta > 0$) such that $2d\, A \sum_{y\in\partial'\Lambda_L} |G^W_{E+i\zeta}(x,y)| < \varepsilon$ Then,

$$|\psi(x)| \;\leq\; \zeta \sum_{y\in\mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \;+\; \varepsilon \tag{90}$$

Now suppose that $\liminf_{\zeta\nearrow 0} \zeta \sum_{y\in\mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| = 0$ Then, as ε was arbitrary (90) implies $\psi(x) = 0$ which is a contradiction. □

We apply the above theorem to our model.

Theorem 7.2. *Assume Condition* (24) *holds and let W be an admissible potential on $\mathcal{X}_\infty$.*

Then, for each $x_0 \in \mathbb{Z}^d$ and for all $E \in \mathcal{E}$

$$\sup_{x\in x_0+\mathcal{C}_0}\ \sup_{\zeta\searrow 0} \sum_{y\in\mathbb{Z}^d} |G^W_{E+i\zeta}(x,y)| \;=\; \infty \tag{91}$$

Proof. Take $E \in \mathcal{E}$, then $E = e_L + \eta$ with

$$\eta \;=\; 2\sum_{k=1}^{d_2} \cos(\pi\kappa_k) \tag{92}$$

for some L and some κ_k. It follows that

$$\psi(x_1,\ldots,x_{d_1},y_1,\ldots,y_{d_2}) \;:=\; \prod_{j=1}^{d_1} \sin(\frac{\pi\ell_j}{p_j}x_j) \prod_{k=1}^{d_2} e^{\mathrm{i}\pi\kappa_\mathrm{k}\mathrm{y}_\mathrm{k}} \tag{93}$$

vanishes on Γ and is a solution to

$$H_0\psi \;=\; \big(H_0+W\big)\psi \;=\; E\,\psi\,, \tag{94}$$

with $||\psi||_\infty \leq 1$. An application of Theorem 7.1 gives the result. □

[15]

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