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Dynamic Contracting for Innovation Under Ambiguity

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Dynamic Contracting For Innovation Under Ambiguity[1]

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Outsourcing of research is commonly observed in knowledge-intensive industries e.g. biotech. We model innovation as an ambiguous stochastic process, and assume that the commercial firms are more ambiguity averse than the research labs. We characterize the optimal sequence of short-term contracts governing innovation, and show how it facilitates ambiguity-sharing. The firm's ambiguity aversion mitigates the dynamic moral hazard problem, resulting in monotonically decreasing investment and prevents equilibrium delay. However, compared to an ambiguity-neutral policymaker's benchmark, the research alliance stops experimenting earlier, and may liquidate the project even after being patented; even redesigning patent laws cannot solve both of the problems.

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1. INTRODUCTION

Outsourcing of research is a large and growing trend in knowledge intensive sectors (e.g. biotechnology, information technology, and software sectors). In these industries, big commercial firms often outsource their research to smaller research oriented firms. These inter-organizational research alliances are generally voluntary agreements between firms involving exchange, sharing or co-development of products, technologies, or services, and play an important role in organizing R&D in the innovation-intensive industries (Baker et al.(2008)). For example, in biotechnology sector, 650 new alliances formed in 2006 alone, with related financial commitments of over $90 billion (Edwards (2007)). During 1996-2007, the industry-university strategic partnerships alone resulted in $457.1 billion worth of patented innovations (Sytch and Philipp (2008)). In pharmaceutical industry, more than 70% of the U.S. companies are involved in research partnerships, and each year on average 25% of the 26bn industry-financed R&D is invested through research alliances (Biopharmaceutical Research Industry Profile (2013)). Information technology sector, accounting for 37% of all strategic research partnerships, registered 254 technology agreements in the year 1996 (Reddy (2001), Hagedoorn et.al.(1992)). In this paper, we study the nature of the dynamic contracts that govern this type of research alliance where the contractees differ in the attitude towards the contextual ambiguity which is inherent in innovation industries.

In the context of innovation, the projects in question are unique in nature. So, sufficient amount of data from very similar situations are generally not available to form a reliable estimate of the true profitability of the project. So it is often difficult to form a unique single-valued probabilistic belief about the true state of nature. Such situations can be modeled as "Knightian uncertainty", or, "ambiguity", using Knight's definition (Knight (1921)):

> "The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation a priori or from statistics of past experience), while in the case of uncertainty this is not true, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique."

In innovation contexts, then, we can assume that the researching entities know only a partial description of the underlying probability distribution associated with the choices. Here, we model innovation as a stochastic ambiguous process, with the research labs, specialized in dealing with ambiguity, are assumed to be less ambiguity averse than the commercial firms. The strategic partnerships between the commercial firms and the research firms aim to exploit the gains from this specialization to deal with ambiguity through the use of dynamic contracting.

More generally, this paper focuses on the optimal sequence of short term contracts that govern the alliance between two parties involved in research activity characterized by contextual ambiguity. We show how their difference in the attitude towards ambiguity leads to an ambiguity sharing agreement, alleviating the dynamic moral hazard problem. This prevents funding delays on the equilibrium path and ensures a monotonically non-increasing funding flow over time. However, for a range of beliefs the alliances choose not to develop the product that they have been patented for, which resembles Patent Troll[3] like cases observed in real life.

While the standing example used in this paper is that of a research alliance, the results also apply to the research cells within a big firm enjoying sufficient autonomy, innovation based departments within a large university, and many such situations commonly observed in these industries.

Following Gilboa and Schmeidler's seminal work on ambiguity (Gilboa and Schmeidler (1989)), multiple prior models of ambiguity have been applied to various decision making contexts. However, in dynamic setting with multiple priors, prior-by-prior updating of belief using Bayes rule leads to dynamic inconsistency. In this paper, we use the ambiguity framework developed in Dumav and Stinchcombe (2013). This framework characterizes a vonNeumann Morgenstern approach to ambiguity and we can use Bayes rule to obtain dynamically consistent updating of beliefs[4].

Our contribution is, therefore, threefold. First, we formulate a model to capture the dynamics of the research partnerships under ambiguity, and characterize the dynamic contract that governs this type of alliances, illustrating the role of ambiguity sharing among the contractees. The optimal contract features some widely observed phenomena in the research intensive industries: (a) no delay in funding, (b) weakly decreasing investment volumes, (c) early dissolution of alliances, and (d) Patent troll like cases. Second, in this environment, we evaluate the research alliances as a mode of organizing research and analyze the role of government policies in such contexts. Lastly, we use a new interpretation of ambiguity due to Dumav and Stinchcombe (2013) and apply it in a dynamic contracting environment[5]. This provides a tractable framework to be used in applied contexts in order to illustrate the effects of ambiguity.

The paper is organized as follows. In the remainder of this section, we motivate the interpretation of ambiguity we are dealing with using a real life example, and place this study in the existing body of related literature. Section 2 contains the model and analyzes the main results of the paper. Also, we consider some generalizations of the model. Section 4 reflects on the general implications of the results. The last section summarizes the findings of this paper and concludes.

[3] A Patent troll is a company or a person that acquires patenting rights without actually developing the goods or services it has been patented for.

[4] Other alternative approaches to modelling ambiguity averse preferences in a dynamically consistent way include the works by Epstein and Schneider (2003), Epstein and Schneider (2007), Maccheroni, Marinacci, and Rustichini (2006), Klibanoff, Marinacci and Mukherji (2009), Siniscalchi (2011), Machina and Schmeidler (1992). For a review, see Machina and Siniscalchi (2014).

[5] However, the intuition of the results derived here will remain the same if we use any other dynamically consistent approach to modeling ambiguity. The approach used here gives a tractable framework to obtain dynamic consistency.

1.1. Motivating Example

Warner-Lambert-Ligand agreement

Let us examine the case of Warner-Lambert & Ligand Agreement (1999) (henceforth W-L): a research, development, and license agreement between Warner-Lambert, a large pharmaceutical company, and Ligand Pharmaceuticals, a much smaller biotech company. This example (a) highlights the contractual features which we will examine in our theoretical model, and more importantly, (b) illustrates how the notion of ambiguity is an important element in such innovation contexts.

The W-L partnership was engaged in a directed research to discover and design small-molecule compounds that act through the estrogen receptors, to develop those compounds into pharmaceutical products, and to take those products through the FDA approval process and then commercialization. They started off with almost 10,000 compounds, out of which only 250 compounds reached the pre-clinical stage. During the research stage, Ligand engaged in directed research, with Warner-Lambert providing the bulk of the funding. The research stage consisted of three periods with duration of fifteen months to three years, after each of the periods Warner-Lambert had the option of unilaterally abandoning the project with little or no direct cost.

Once a successful compound was identified, the project moved from the research to the development stage, and regulatory and market experience became more important. The cost of the project, all of which will be borne by Warner-Lambert, also increased exponentially. As a result, both responsibility and decision making shifted to Warner-Lambert, who had the option to develop the project.

- *Innovation as an Ambiguous Process:*

  In this strategic partnership between Warner-Lambert and Ligand, the research could have ended in one of the three possible ways:

  (a) They could have found a molecule which passes all the clinical trials and is found fit to be developed into a drug. This can be modeled as the case when the true state (or, profitability) of the project is *Good*.

  (b) They could have failed to find a suitable molecule that acts through the estrogen receptors, after testing all the candidate molecules. This case can be modeled as the true state being *Bad*.

  (c) Apart from these two states, the research could have ended in finding one (or more) molecule which is capable to work through the estrogen receptors, but, given the state of the current pharmaceutical technology, can not be developed into a drug. If the research finds such a molecule, it is not presently known if in the future a new pharmaceutical technology will be invented so that the molecule(s) can be developed into a drug; or if there is some chemical property of the molecule(s) because of which it (they) can never be processed as a drug. So, in this case, even after conducting the decade-long research, it is possible stumble upon "open questions". We model this state as a new epistemic state and call it *Unknowable* or *Amalgamated,* because if the research ends up here, the innovation process is not yet understood. The existence of such unknowable states distinguish research activities from other manufacturing activities where the production process is well understood. In the framework of ambiguity we are dealing with in this paper (Dumav and Stinchcombe (2013)[6]), this new epistemic state captures the essence of ambiguity.

Notice that for Ligand, which is a research based firm, reaching this state is still valuable, because it adds to their stock of intellectual capital and leads to further research into the properties of these molecules, but Warner-Lambert will not value this state as much. So, it is natural to assume that Ligand is less ambiguity averse than W-L.

Next, we note the specifics of this contract.

Contractual Features:

- *Short Term Contracting:* In W-L agreement, each contracting phase lasted for fifteen months up to three years, whereas the partnership lasted for more than a decade. Similarly, many of the collaborative R&D ventures are governed by short term contracts, with the contracting terms being renegotiated after every contracting phase. This paper studies the optimal sequence of short term contracts without commitment.

- *Rich forms of collaborating:* The W-L agreement contained a rich braiding of explicit (legally enforceable) and implicit (legally unenforceable) terms (Gilson, Sabel and Scott (2009)). On one hand there was an elaborate description of the payments under various possible contingencies (e.g., the milestone bonuses, the royalty rate), which are legally enforceable. On the other hand, the contract specifies the control rights and property rights, which gives unilateral decision power to one of the contracting parties[7]. To mimic this complex contracting structure, the present model allows for both a state contingent payment structure and a unilateral liquidation power.

- *Moral Hazard:* Since Warner-Lambert could not perfectly monitor Ligand's activity, it was possible for Ligand to divert the resources (time and money) in order to either cross-subsidize other projects or for personal gain. Such cross-subsidization

[6] Appendix B contains the preliminaries of this framework.

[7] The gap between contract formation and the appearance of a marketable drug was more than a decade. So, Ligand's compensation was carefully structured. First, it was paid for some fraction (perhaps all) of the resources assigned to the task. Second, the agreement established a number of specific *milestones*, and, upon reaching each milestone, Ligand received an additional payment. Finally, after the research produced marketable products, Ligand received royalty payments on sales. However, if Warner-Lambert chose to abort the project at any time, they retained the property rights.

possibility gives rise to potential moral hazard problem in such contractual relationships.

In the dynamic relationship between the two firms, the moral hazard problem is more severe. Apart from the one-time gain by diverting resources, the researching party can also appropriate a dynamic gain from diversion, since following a diversion of resources, the learning paths for the two firms diverge. This gives rise to a further incentive to cheat and is referred to in the literature as the "dynamic moral hazard" problem. In this model we consider a dynamic contracting environment, taking into account this potential dynamic moral hazard problem.

In the innovation-intensive sectors we can find many such examples, where we can motivate the existence of this new epistemic state: "Unknowable" and consider innovation to be an ambiguous process. This paper uses this interpretation of ambiguity and examines the contractual structure that governs the inter-organizational research partnerships under this type of uncertainty.

1.2. Related Literature

This paper adds to the literature on *optimal contracts for experimentation.* It is most closely related to Bergemann and Hege (1998) and Bergemann and Hege (2005), which characterize the optimal contract for experimentation modeling innovation as a risky stochastic optimal stopping problem. In this framework, they document the potential dynamic moral hazard problem and how it leads to possible in-equilibrium delay of funding (in finite horizon) and monotonically increasing investment volume in infinite horizon. Hörner and Samuelson (2013) examine a similar framework of experimentation in continuous time and characterize all possible equilibria.

There are two significant differences between these papers and ours. Firstly, here we consider innovation as an ambiguous process, rather than a risky one. We show that the introduction of ambiguity and the different attitudes towards ambiguity among the contractees alleviate the dynamic moral hazard problem, preventing in-equilibrium delay in funding in the finite horizon case, and in the infinite horizon this leads to a monotonically decreasing level of investment. Also, we use non-conclusive signals in the experimentation stage, which gives rise to a positive option value of waiting and changes the optimal contract structure. It illustrates the role of patent laws, which enables us to evaluate the alliances as a mode of organizing research, and analyze the role of government policies in innovation.

Optimal contracting for experimentation under moral hazard or adverse selection concerns has been studied in a vast body of literature. Bonatti and Horner (2009) and Campbell et al.(2013) study experimentation in teams, with unobservable actions and they also find the possibility of financing delay. Lerner and Malmendier (2010) show how incomplete contracts can be used as optimal contractual design to solve the problem of moral hazard in biotechnology research partnerships. Poblete and Spulber (2014) discuss the optimal design of delegated experimentation and shows that the optimal sequential search in this case involves a stopping rule, so the optimal contract turns out to be an option. In different contexts Mason and Välimäki (2015), Akcigit and Liu (2015), Green and Taylor (2016), Halac, Kartik, and Liu (2016), Adrian and Westerfield (2009), He et al.(2017), Manso (2011), Ederer and Manso (2013) , Jeitschko and Mirman (2002), Jeitschko et al.(2002) all analyze different versions of the contracting problem to motivate innovation[8]. Our paper joins this line of enquiry with the additional assumptions of ambiguity and no commitment power. In this environment, we find the optimal sequence of short term contracts and discuss the policy implications.

This paper also fits in a vast and growing strand of literature on *dynamic contracting problems* in discrete and continuous times. Bergemann and Pavan (2015) contains a detailed survey of this literature. The importance of dynamic agency cost has been well documented in literature using both the continuous time framework (DeMarzo and Sannikov (2016), Sannikov (2008)) and discrete time models (Bhaskar and Mailath (2017), Bhaskar (2012)). In the present study, we show that the presence of ambiguity and difference in attitude towards ambiguity among the contracting parties alleviate the dynamic moral hazard problem.

Lastly, this paper is also related to the literature on *ambiguity in a contracting environment*. Besanko, Tong and Wu (2012) considers innovation as an ambiguous process and examines contracting for delegated experimentation with unobservable agent type. In contrast, in this paper we study moral hazard rather than adverse selection. Also, in contrast to some other related papers that analyze dynamic contracting problems in non-standard choice theoretic setting (e.g. Szydlowski (2012). , Dumav and Riedel (2014)), we use a novel approach to ambiguity: in our framework we can use Bayesian updating instead of maximum likelihood updating to ensure dynamic consistency[9].

2. MODEL AND ANALYSIS

2.1. The Model

States: The innovation activity is centered around a project, success of which depends on the true state or true profitability of the project: $\theta \in \Theta$. The true state is not known; moreover, it is not possible to form a single probabilistic assessment about it. In a multiple prior setting, the state space and the common prior belief shared by

[8] Apart from these papers, there is an entire strand of literature that models experimentation using two-armed bandit framework, starting with Keller, Rady, and Cripps (2005). For a review of that literature, refer to Hörner and Skrzypacz (2016)..

[9] There is another prominent strand of literature on mechansim design and contracting problems under ambiguity in a static context: see Lopomo, Rigotti and Shannon (2011),Garrett (2014), Carroll (2015), Amarante, Ghossoub and Phelps (2012). For a detailed survey, refer to Mukerji and Tallon (2004) and Etner, Jeleva and Tallon (2012).

all the agents can be represented as:

$$\begin{aligned} \Theta &= \{Good, Bad\} \\ P(\theta &= Good) = [r_0, s_0] \;\; ; 0 \leq r_0 < s_0 \leq 1. \end{aligned}$$

Using the framework of ambiguity developed in Dumav and Stinchcombe (2013)[10], we observe that the class of non-empty, closed, convex subsets of probabilities like $[r_0, s_0]$ can be represented as a simplex:

$$\boldsymbol{K}_{\Delta(\Theta)} = \{[r, s] : 0 \leq r \leq s \leq 1\}$$

Thus, this multiple prior $[r_0, s_0]$ has a unique representation as a convex combination of extreme states given by $\Theta' = \{Good, Bad, Unknowable\}$, where the new epistemic state "*Unknowable*" is motivated in the previous example.

Each $[r_0, s_0]$ has a unique representation as:

$$[r_0, s_0] = r_0[1, 1] + (1 - s_0)[0, 0] + (s_0 - r_0)[0, 1]$$

where the state *Unknowable* is represented as $[0, 1]$; the state at which the decision maker knows only that the probability of $\theta = Good$ is someplace between 0 and 1.

Thus, in this framework, we can alternatively represent this set-valued prior by a three state expected utility model, where the true state of the project lies in Θ' :

$$\begin{aligned} \Theta' &= \{Good, Bad, Unknowable\} \\ P(\theta &= Good) = r_0 \\ P(\theta &= Bad) = 1 - s_0 \\ P(\theta &= Unknowable) = s_0 - r_0 \\ 0 &\leq r_0 < s_0 \leq 1. \end{aligned}$$

That is, with probability r_0, at the end it will be revealed that the project is profitable, with probability $1 - s_0$ it will be revealed that the project is not profitable, but with probability $s_0 - r_0$, the true profitability of the project will turn out to be "*Unknowable*", or, Not Yet Known, depending on the current state of technology and knowledge. Notice that $s_0 - r_0$ captures the idea that the decision maker knows only a partial description about the underlying distribution; if $r_0 = s_0$ then we are back to the "risky" situation.

If the payoff for $\theta = Good$ is u_G, for $\theta = Bad$ is $u_B < u_G$, then the payoff associated with the new state $\theta = Unknowable$ is computed as:

$$u(\theta = Unknowable) = \frac{1}{2}(u_G + u_B) - \frac{v}{2}(u_G - u_B);$$

[10]Refer to the Representation Theorem 1 in Appendix B.

where the ambiguity aversion parameter v captures the attitude towards ambiguity. $v > 0$ refers to the decision maker being ambiguity averse. The higher v is, the more the decision maker dislikes the state $\theta = Unknowable$, hence can be considered as more ambiguity averse. Here, I assume $v \in [0, 1]$.

Contracting Parties:

The two parties forming the research alliance are: a big commercial firm (henceforth CF) and the smaller research-oriented firm or research lab (henceforth RL). We assume that both the parties are risk-neutral and share a common prior about the true profitability of the project. RL is ambiguity neutral while CF is ambiguity averse[11].

Also assume that the contracting parties do not have the power to commit to a long term contract, hence innovation occurs in a sequence of short term contracts.

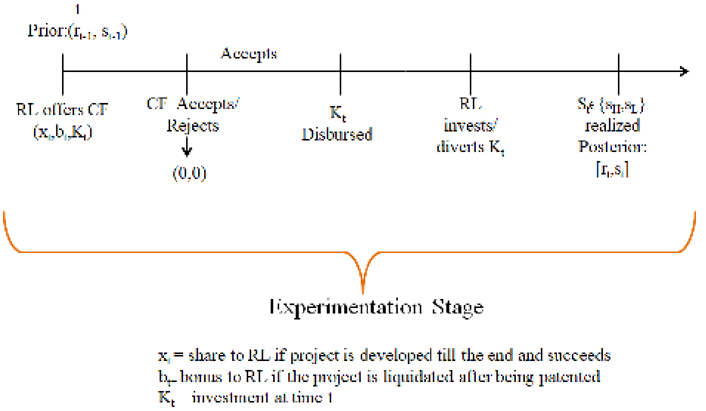

FIG. 1 Contracting Time Line: Experimentation Stage

Contracting Time Line: RL owns the project, but is liquidity constrained, so CF funds the project. The contracting time line involves two stages: an experimentation stage and a development stage, as depicted in the figures 1 and 2 below.

Experimentation Stage: In the experimentation stage, at the beginning of each period t, RL makes a take-it-or-leave-it offer[12] to CF specifying

(a) $x_{t:}$: the proportional share of the final return RL receives, if the project is developed till the end

(b) b_t : the bonus that RL gets once the project is granted a Patent, and,

(c) K_t : amount of investment to be disbursed in the t^{th} period.

CF accepts or rejects the offer. If accepted, the funds are disbursed and then RL privately decides whether to invest the fund or divert it for personal benefit (or cross-

[11]Essentially, we are assuming that it is not possible to contract the know-how derived from working on a project. The ability to write a contract on the knowledge generated from the research can change the ambiguity attitude of the two firms.

[12]Here, it is assumed that the research lab faces a competetive market of commercial firms for that project, hence enjoys all the bargaining power. This is a simplifying assumption. Relaxing the full bargaining power assumption does not qualitatively change the results.

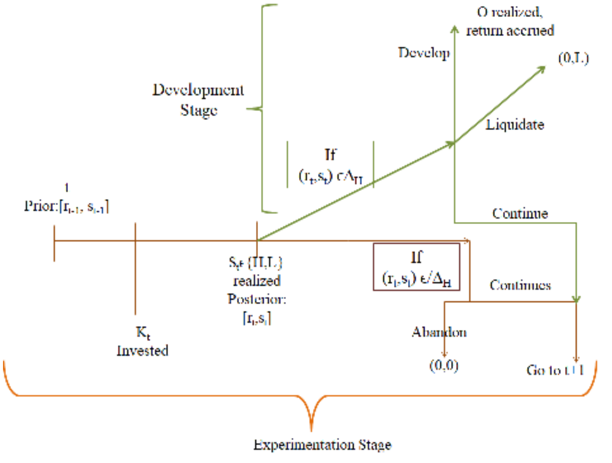

FIG. 2 Contracting Time Line: Development Stage

subsidization). At the end of the period, a binary signal S_t is publicly realized and beliefs are accordingly updated from $[r_{t-1}, s_{t-1}]$ to $[r_t, s_t]$.

Development Stage: Now, assume that there is an exogenously given patenting threshold in place which the research alliance has to obey. For the purpose of this model, this threshold is in terms of posterior belief after observing the signal that comes out of the experimentation: the project is granted a patent if the posterior belief after observing the public signal at t^{th} *period* : $[r_t, s_t] \in \Delta_H \subseteq \boldsymbol{K}_{\Delta(\Theta)}$ (refer to Figure). In Section 3, we will analyze how a non-strategic policymaker may endogenously determine this threshold.

Once the posterior belief after observing the signal clears the patent threshold, i.e., $[r_t, s_t] \in \Delta_H$, the project is allowed to move to the Development Stage. In the Development stage, CF unilaterally decides whether to continue developing the product, liquidate the project, or keep experimenting further:

$$a_t^{CF} \in \{Develop, Liquidate, Continue\}$$

If the project is continued till the end, after investing the fixed amount I, the true state θ is realized and returns accrue to the contracting parties. If the project is liquidated, CF appropriates the property rights, therefore obtains the liquidation value L minus the bonus b_t.

If the posterior belief is not high enough , i.e., $[r_t, s_t] \notin \Delta_H$, then CF decides whether to continue experimenting at period $t + 1$ with updated beliefs, or to abandon the project, earning a return of 0 forever:

$$a_t^{CF} \in \{Abandon, Continue\}$$

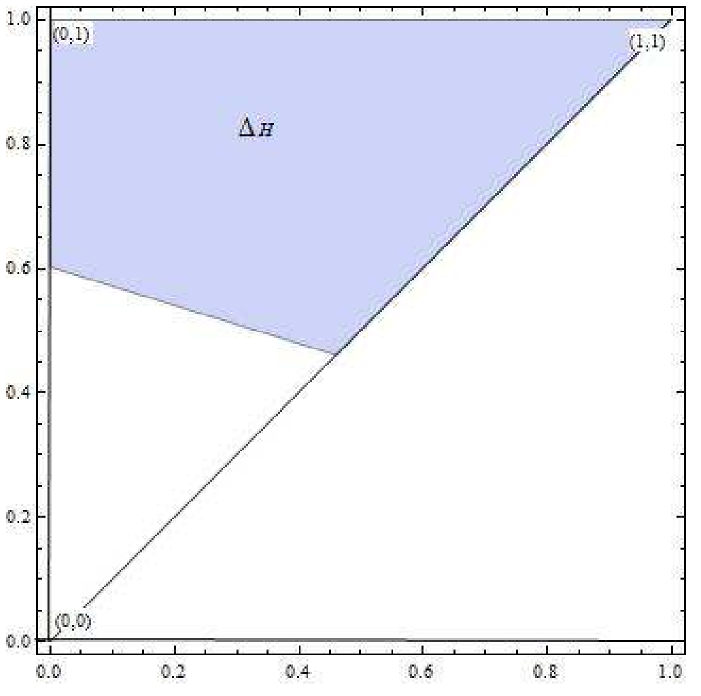

FIG. 3 Patent Law

Right now, we only assume that any reasonable patent rule has to ensure that the expected return from developing the project has to be at least as high as the liquidation value.

$$\Delta_H := \left\{ [r_t, s_t] \subseteq \boldsymbol{K}_{\Delta(\Theta)} | \frac{r_t + s_t}{2} R - I \geq L \right\} \tag{1}$$

Assume that the parameters are such that:

$$\begin{aligned} R &> L > \overline{K} \\ R &> 2\overline{K}(I + L) \\ L &< 2\overline{K} \end{aligned} \tag{2}$$

Signal Structure: The public signal observed at the end of every period during the experimentation stage is binary and conditionally independent, they are informative about the true state $\theta \in \Theta'$[13] . Also, assume that at any period t, investment flow increases signal precision.

$$P(S_t = s_H|\theta) = \lambda_\theta K_t; \theta \in \{G, B, U\}$$

[13] Intuitively, the signals can be thought of as random draws from a Bernoulli distribution: $S_t \sim Bernoulli(\lambda_\theta K_t)$ if $\theta = \{G, B, U\}$

where

$$1 > \lambda_G K_t \geq \lambda_U K_t \geq \lambda_B K_t \geq 0 \quad \forall K_t \in [0, \overline{K}] \tag{A1}$$

In this linear parametric structure, this assumption ensures that (a) $\lambda_\theta(K_t)$ is a valid probability measure defined on Θ', (b) higher investment increases the signal precision. This is similar to the Monotone Likelihood Ratio Property[14].

The conditional distribution associated with this binary signal is characterized below:

S_t	s_H	s_L
$\theta = G\ (1,1)$	$r_{t-1}\lambda_G K_t$	$r_{t-1}(1-\lambda_G K_t)$
$\theta = Unknowable\ (0,1)$	$(s_{t-1}-r_{t-1})\lambda_U K_t$	$(s_{t-1}-r_{t-1})(1-\lambda_U K_t)$
$\theta = B(0,0)$	$(1-s_{t-1})\lambda_B K_t$	$(1-s_{t-1})(1-\lambda_B K_t)$
	μ_t	$1-\mu_t$

So that, at any t^{th} period,

$$\begin{aligned} P(S_t &= s_H) = \mu_t(K_t) = K_t\lambda_t \\ &= K_t\underbrace{[r_{t-1}\lambda_G + (1-s_{t-1})\lambda_B + (s_{t-1}-r_{t-1})\lambda_U]}_{\lambda_t} \end{aligned}$$

After observing the binary signal, at the end of each period, the beliefs are updated using Bayes Rule. For example, after observing a high signal $S_t = s_H$, the updated posterior on the true state being Good is as follows:

$$P(\theta = G | S_t = s_H) = \frac{r_{t-1}\lambda_G}{\mu_t} = r_t^H$$

To save on notation, let us define the average of the posterior belief as the posterior mean and the average spread of the posterior belief as the posterior ambiguity:

$$\begin{aligned} posterior\ mean &= \frac{r_t + s_t}{2} = p_t \\ posterior\ ambiguity &= \frac{s_t - r_t}{2} = q_t \end{aligned}$$

Note that, by $MLRP$, after observing $S_t = s_H$, posterior mean p_t increases and posterior ambiguity q_t decreases; and after $S_t = s_L$, p_t decreases and q_t increases.

2.2. Two Period Example

Before discussing the infinite horizon model, let us first analyze the two period contracting game in order to illustrate the intuitions behind the main results of this paper. The findings from this two period example are readily extendable to a finite horizon contracting problem, and they will provide the intuitive understanding about the model in the general infinite horizon setting.

[14] Section 4 discusses the case with a non-linear signal structure that retains this MLRP.

Here, let's assume that the project is exogenously terminated after $t = 2$.

First of all, note that since RL has full bargaining power, the participation constraint of CF holds as an equality in every period. So, whenever the posterior belief enters the patenting region, experimentation stops.

So,

$$\Pr((r_2, s_2) \in \Delta_H) = \Pr(S_2 = s_H) = \mu_2 = K_2\lambda_2$$

RL solves:

$$V_2(r_1, s_1) = \max_{a_2^{CF}} \{V_2^{Dev}, V_2^{Liq}\} \tag{3}$$

where

$V_2^{Dev} =$ RL's expected payoff from period 2 if, given the contractual terms, CF develops the product after reaching $\Delta_{H}.(a_2^{CF} = Dev)$

$V_2^{Liq} =$ $RL's$ expected payoff from period 2 if, given the contractual terms, CF liquidates the product after reaching $\Delta_{H}.(a_2^{CF} = Liq)$

$$V_2^{Dev} = \max_{x_2, b_2, K_2} \mu_2 [Rp_2|_{(r_2,s_2)\in\Delta_H} x_2]$$

$$\mu_2 [Rp_2|_{(r_2,s_2)\in\Delta_H} x_2] \geq K_2 \qquad (IC_{2,Dev}^{RL})$$

$$\mu_2 [R(p_2 - vq_2)|_{(r_2,s_2)\in\Delta_H} (1 - x_2) - I] \geq K_2 \qquad (PC_{2,Dev}^{CF})$$

$$R(p_2 - vq_2)|_{(r_2,s_2)\in\Delta_H} (1 - x_2) - I \geq L - b_2 \qquad (IC_{2,Dev}^{CF})$$

$$x_2 \in [0, 1]; b_2 \geq 0; K_2 \in [0, \overline{K}]$$

$$V_2^{Liq} = \max_{x_2, b_2, K_2} \mu_2 [b_2]$$

$$\mu_2 [b_2] \geq K_2 \qquad IC_{2,Liq}^{RL}$$

$$\mu_2 [L - b_2] \geq K_2 \qquad (PC_{2,Liq}^{CF})$$

$$R(p_2 - vq_2)|_{(r_2,s_2)\in\Delta_H} (1 - x_2) - I \leq L - b_2 \qquad (IC_{2,Liq}^{CF})$$

$$x_2 \in [0, 1]; b_2 \geq 0; K_2 \in [0, \overline{K}]$$

Let us take a closer look at the constraint set. In each of the cases, IC_2^{RL} s are the standard incentive compatibility constraints for RL, ensuring no diversion of funds on the equilibrium path. Notice that, because of the linear structure, if any partial diversion is beneficial, so is the full diversion, that is why it is sufficient to consider the incentive constraint only for the full diversion case. $PC_{2,}^{CF}$ s are the participation constraints for CF, guaranteeing CF an expected return to cover the investment cost. Without loss of generality, $CF's$ outside option is normalized to 0. The last constraint ensures that after the signal realization, it is sequentially optimal for CF to develop the project in the first case and liquidate in the second.

Solving 3, we get two regions of posterior belief: Δ_D, Δ_L:

$$\begin{aligned}\Delta_D &= \{(r_2, s_2) \in \Delta_H |\ a_2^{CF}|_{\Delta_H} = Dev \\ &\quad i.e., CF \text{ chooses to develop the project once being granted a patent}\} \\ \Delta_L &= \{(r_2, s_2) \in \Delta_H |\ a_t^{CF}|_{\Delta_H} = Liq \\ &\quad i.e., CF \text{ chooses to liquidate the project once being granted a patent}\}\end{aligned}$$

The payment rules are:

if $(r_2, s_2) \in \Delta_D$	if $(r_2, s_2) \in \Delta_L$
$x_2 = 1 - \frac{1}{R(p_2 - vq_2)}\left(I + \frac{1}{\lambda_2}\right)$;	$b_2 = L - \frac{1}{\lambda_2}$
$b_2 \geq L - \frac{1}{\lambda_2}$	$x_2 \geq 1 - \frac{1}{R(p_2 - vq_2)}\left(I + \frac{1}{\lambda_2}\right)$
$K_2^{Dev} = \overline{K} = K_2^{Liq}$	

And the expected value to RL from $t = 2$ is:

$$V_2(r_1, s_1) = \lambda_2 \overline{K} \max\left\{p_2\left(R - \frac{1}{p_2 - vq_2}\left(I + \frac{1}{\lambda_2}\right)\right), \left(L - \frac{1}{\lambda_2}\right)\right\} \tag{4}$$

Now, let us go one step backward at $t = 1$.

At $t = 1$, RL solves:

$$V_1(r_0, s_0) = \max_{a_1^{CF}}\{V_1^{Dev}, V_1^{Liq}\}$$

where V_1^{Dev} and V_1^{Liq} are defined as in $t = 2$, with three sets of constraints: an incentive constraint for RL ensuring no diversion, a participation constraint for CF requiring that they continue investing, and an incentive constraint specifying that it is indeed beneficial for CF to Develop in Δ_D and Liquidate in Δ_L. In period 1, compared to the problem at $t = 2$, the participation constraint and incentive constraint for CF remain the same with the corresponding posterior belief at $t = 1$; however the incentive constraint for RL requires a closer look. The incentive constraints $(IC_{1,Liq}^{RL})$ and $(IC_{1,Dev}^{RL})$ highlight the two sources of gain from cheating: the static gain and the dynamic gain.

$$\begin{aligned}&K_1\lambda_1[Rp_1|_{(r_1,s_1)\in\Delta_H} x_1] + \delta(1 - K_1\lambda_1)E_1V_2(r_1, s_1|_{(r_1,s_1)\notin\Delta_H}) \\ &\qquad \geq K_1 + \delta E_1 V_2(r_1, s_1; r_0, s_0|_{(r_1,s_1)\notin\Delta_H}) \qquad (IC_{1,Dev}^{RL}) \\ &K_1\lambda_1 b_1 + \delta(1 - K_1\lambda_1)E_1V_2(r_1, s_1|_{(r_1,s_1)\notin\Delta_H}) \\ &\qquad \geq K_1 + \delta E_1 V_2(r_1, s_1; r_0, s_0|_{(r_1,s_1)\notin\Delta_H}) \qquad (IC_{1,Liq}^{RL})\end{aligned}$$

The static gain is similar as in the second period, stemming from the benefit RL derives by diverting the investment amount (K_1), so the IC at $t = 1$ has to ensure that $RL's$ expected payoff from $t = 1$ has to be greater than the investment. However, there is a dynamic gain from cheating as well, captured by the dynamic cheating value: which arises from the fact that following a diversion of funds at $t = 1$, the posterior belief of RL and CF diverge. Because of the diversion, the signal S_1 is always s_L, observing which CF is prompted to update his belief to $[r_1, s_1]|_{S_1=s_L}$, with posterior mean p_1 and ambiguity q_1. The next period's contract will then be based on this public belief $[r_1, s_1]$. However, RL has perfectly observed his own action, so even after the low signal he does not update his belief and evaluates the future contracting terms using his private belief $[r_0, s_0]$. This constitutes the **dynamic agency cost**:

$$\begin{aligned}
DAC_2 &= \delta[V_2(cheat) - V_2(no\ cheat)] \\
&= \delta[E_1 V_2(r_1, s_1; r_0, s_0) - (1 - K_1\lambda_1)E_1 V_2(r_1, s_1)] \\
&= \begin{cases} \delta\left[\frac{\lambda_1 p_1}{\lambda_2 p_2} - (1 - K_1\lambda_1)\right] V_2(r_1, s_1) & \text{if } (r_2, s_2) \in \Delta_D \\ \delta\left[\frac{\lambda_1}{\lambda_2} - (1 - K_1\lambda_1)\right] V_2(r_1, s_1) & \text{if } (r_2, s_2) \in \Delta_L \end{cases} \\
&> 0
\end{aligned}$$

This dynamic agency cost may lead to delay in funding for a range of parameter values for which IC_2^{RL} is satisfied but not IC_1^{RL} :

$$\Delta_{Delay} := \left\{(r_1, s_1)|\max\{\mu_2 Rp_2, \mu_2 b_2\} \geq \bar{K} > \max\{\mu_1 Rp_1, \mu_1 b_1\} - DAC_2\right\} \quad (5)$$

Under risk, $r_t = s_t$ for all t, and it can be seen that $\Delta_{Delay} \neq \phi$. The possibility of in-equilibrium delay due to dynamic agency cost is well documented in the literature of dynamic contracts (Bergemann and Hege (1998); Bonatti and Horner (2011)). However, in the present scenario with ambiguity, we find that the dynamic agency cost, and consequently the region Δ_{Delay} where delay happens shrinks as ambiguity aversion of CF goes up. In fact, if $CF's$ ambiguity aversion is higher than a threshold, then equilibrium delay never happens whenever the contracting parties are not infinitely patient. In the presence of ambiguity, due to the ambiguity sharing contractual arrangement, the commercial firm's ambiguity aversion reins in the dynamic moral hazard problem.

Intuitively, CF, being ambiguity averse, becomes much more cautious and pessimistic after each low signal. So, following a low signal, CF has to be guaranteed a greater share of the final return in order to keep investing. This ambiguity sharing agreement disciplines RL and lowers his dynamic expected value from cheating (DAC_2) which, in turn, eases the funding constraint at $t = 1$ and possibility of in-equilibrium delay decreases.

Proposition 1. *For discount rate* $\delta \leq \overline{\delta} = 1 - \left(\frac{p_2 - vq_2}{p_1 - vq_1}\right)^2 \frac{q_1}{q_2}\left(\frac{\frac{1}{\lambda_1}+I}{\frac{1}{\lambda_2}+I}\right)$, $\exists \widetilde{v} \in (0,1)$, *such that* $\forall v \geq \widetilde{v}$, $\Delta_{Delay} = \phi$, *that is, in-equilibrium delay never happens.*

Proof. In Appendix A. ∎

Let us summarize the findings from this two period model. The results qualitatively hold true for all finite $t > 2$ as well.

Ambiguity sharing: Contract effectively shares ambiguity: as CF becomes more ambiguity averse, his share $1 - x_t$ increases.

Alleviation of dynamic moral hazard Ambiguity sharing alleviates dynamic moral hazard problem. In a finite horizon setting, under a range of parameter values, this results in the impossibility of an in-equilibrium delay.

Evolution of share Every period the posterior belief about the project falls. To compensate, CF needs a higher share to continue funding the project ($1-x_t$ increases, hence x_t decreases with t).

Patent Troll If the project clears the patenting threshold after a lot of failed trials, the posterior belief of CF decreases beyond a threshold so that even though the project obtains the patent rights, CF decides not to develop it further: this resembles patent troll like situation. In absence of ambiguity, this never happens because $\forall (r_t, s_t) \in \Delta_H, p_2R - I > L$,but for every $v > 0$, there are $(r_t, s_t) \in \Delta_H$ such that $p_2(1-v)R - I \leq L$.

We define this region of beliefs (Δ_L) as Patent troll region:

$$\Delta_L(v) = \{(r_t, s_t) | ((r_t, s_t) \in \Delta_H) \cap (p_2(1-v)R - I \leq L)\}$$

2.3. Infinite Horizon Model

In this section we formally set up the infinite horizon sequential contracting game between CF and RL and derive the equilibrium contractual outcome. Let us first define the equilibrium.

At any period t, let H_t^P denote the set of all possible public histories up to, but not including, period t. Each element $h_t^P \in H_t^P$ contains

(a) past contractual terms: $\{x_j, b_j, K_j\}_{j=1}^{t-1}$

(b) past strategic choices of CF to accept or reject the contract offered at each period: $\{\zeta_j\}_{j=1}^{t-1}$ ($\zeta_t = 1$ if CF accepts an offer at period t, 0 otherwise)

(c) past realized values of the signals: $\{S_j\}_{j=1}^{t-1}$

(d) past strategic choices of CF after observing the signal realizations at every period: $\{a_j^{CF}\}_{j=1}^{t-1}$.

In contrast, the set of possible private histories is denoted by H_t, which can be potentially different than the private history of RL; who observes his own decision to divert the fund as well. So each element $h_t \in H_t$, in addition to h_t^P, contains $\{d_j\}_{j=1}^{t-1}$, the past realizations of the strategic choices of RL whether to divert the fund ($d_t = 1$ if the fund is invested in period t and 0 if diverted[15]).

The true history leads to the posterior belief formed by RL at the beginning of period t :

$$[r_{t-1}, s_{t-1}] : H_t \rightarrow \mathbb{K}_{\Delta_{[0,1]}}$$

[15] Notice that the linear signal structure implies that if partial diversion of funds is optimal, so is full diversion, so the action set is essentially binary.

In consequence, CF also has a belief about the true history, captured by the belief about the true posterior formed by CF :

$$[r'_{t-1}, s'_{t-1}] : H_t^P \times D'_t \rightarrow \mathbb{K}_{\Delta_{[0,1]}}$$

which depends on the public history as well as the belief CF has about $RL's$ past investment behavior: $\{d'_j\}_{j=1}^{t-1}$. D'_t contains the set of all beliefs $\{d'_j\}_{j=1}^{t-1}$.

Then, a contract (x_t, b_t, K_t) by RL is a mapping from the true history H_t into the sharing rule x_t , bonus rule b_t and investment flow K_t.

$$\begin{aligned} x_t &: H_t \rightarrow [0,1] \\ b_t &: H_t \rightarrow \mathbb{R}_+ \\ K_t &: H_t \rightarrow [0,\overline{K}] \subset [0,1] \end{aligned}$$

A decision rule by CF whether to accept or reject the contract is then a mapping from the perceived history: $\{x_j, b_j, K_j, \zeta_j, a_j^{CF}, d'_j\}_{j=1}^{t-1}$, and the contract proposed, into a binary decision to reject or accept the contract:

$$\zeta_t : H_t^P \times [0,1] \times \mathbb{R}_+ \times [0,\overline{K}] \rightarrow \{0,1\}$$

An investment policy by RL is:

$$d_t : H_t \times [0,1] \times \mathbb{R}_+ \times [0,\overline{K}] \times \{0,1\} \rightarrow \{0,1\}$$

A decision rule by CF after observing the signal at the end of period t is a mapping from the public history, contractual terms, perceived belief about diversion strategy of RL given the incentives provided by the contract, and the realized signal $S_t \in \{s_H, s_L\}$ into the choice to develop, liquidate, continue, or abandon the project at the end of period t.

$$a_t^{CF} : H_t^P \times [0,1] \times \mathbb{R}_+ \times [0,\overline{K}] \times \{0,1\} \times \mathbb{K}_{\Delta_{[0,1]}} \rightarrow \{Dev, Liq, Abandon, Cont\}$$

In this model, we are in a Markovian world, because all the payoff relevant history can be captured by the four state variables: $(r_{t-1}, s_{t-1}, r'_{t-1}, s'_{t-1})$: the true posterior belief held by RL : $[r_{t-1}, s_{t-1}]$ and the belief of CF about the true posterior:$[r'_{t-1}, s'_{t-1}]$. In this context, let us define the suitable Markov equilibrium concept.

Definition 1 (Markov Sequential Equilibrium). A Markov sequential equilibrium is a sequential equilibrium $\{x_t, b_t, K_t, \zeta_t, a_t^{CF}, d_t\}_{t=1}^{\infty}$, if

$$\begin{array}{rcc}
(r_{t-1},s_{t-1})(h_t)=(r_{t-1},s_{t-1})(\hat{h}_t) & \Longrightarrow & \begin{array}{c} x_t(h_t)=x_t(\hat{h}_t) \\ b_t(h_t)=b_t(\hat{h}_t) \\ K_t(h_t)=K_t(\hat{h}_t) \end{array} \\
\left.\begin{array}{c} (r'_{t-1},s'_{t-1})(h_t^P)=(r'_{t-1},s'_{t-1})(\hat{h}_t^P) \\ (x_t,b_t,K_t)=(\hat{x}_t,\hat{b}_t,\hat{K}_t) \end{array}\right\} & \Longrightarrow & \begin{array}{c} \zeta_t(h_t^P,x_t,b_t,K_t) \\ =\zeta_t(\hat{h}_t^P,\hat{x}_t,\hat{b}_t,\hat{K}_t) \end{array} \\
\left.\begin{array}{c} (r_{t-1},s_{t-1})(h_t)=(r_{t-1},s_{t-1})(\hat{h}_t) \\ (x_t,b_t,K_t)=(\hat{x}_t,\hat{b}_t,\hat{K}_t) \\ \zeta_t=\hat{\zeta}_t \end{array}\right\} & \Longrightarrow & \begin{array}{c} d_t(h_t,x_t,b_t,K_t,\zeta_t) \\ =d_t(\hat{h}_t,\hat{x}_t,\hat{b}_t,\hat{K}_t,\hat{\zeta}_t) \end{array} \\
\left.\begin{array}{c} (r_{t-1},s_{t-1})(h_t)=(r_{t-1},s_{t-1})(\hat{h}_t) \\ (x_t,b_t,K_t)=(\hat{x}_t,\hat{b}_t,\hat{K}_t) \\ \zeta_t=\hat{\zeta}_t \\ d_t=\hat{d}_t \end{array}\right\} & \Longrightarrow & \begin{array}{c} a_t^{CF}(h_t^P,x_t,b_t,K_t,\zeta_t,d_t) \\ =a_t^{CF}(\hat{h}_t^P,\hat{x}_t,\hat{b}_t,\hat{K}_t,\hat{\zeta}_t,\hat{d}_t) \end{array}
\end{array}$$

$$\forall h_t\in H_t;\forall h_t^P\in H_t^P;\forall\hat{h}_t\in\hat{H}_t;\forall\hat{h}_t^P\in\hat{H}_t^P;\forall(x_t,b_t,K_t),(\hat{x}_t,\hat{b}_t,\hat{K}_t);\forall\zeta_t,\hat{\zeta}_t,\forall d_t,\hat{d}_t$$

The Markovian sequential equilibrium ensures that the continuation strategies are time consistent and identical after any history with identical updated true posterior belief $[r_{t-1},s_{t-1}]$ and $CF's$ belief about the posterior: $[r'_{t-1},s'_{t-1}]$. It imposes that on the equilibrium path CF has the true belief given the incentives, i. e., on the equilibrium path $[r_{t-1},s_{t-1}]=[r'_{t-1},s'_{t-1}]$, but allows for the possibility of divergence of posterior beliefs off the equilibrium path.

The stopping regions are defined as before:

$$\begin{aligned}
\Delta_D &= \{(r_t,s_t)\in\Delta_H|\ a_t^{CF}=Dev\} \\
\Delta_L &= \{(r_t,s_t)\in\Delta_H|\ a_t^{CF}=Liq\} \\
\Delta_S^C &= \{(r_t,s_t)\in\mathbb{K}_{\Delta_{[0,1]}}|\ a_t^{CF}=Abandon\}
\end{aligned}$$

Now, at every period t, RL solves:

$$\begin{aligned}
&V_t(r_{t-1},s_{t-1}) \\
&= \begin{array}{c} \max_{\Delta_D,\Delta_L,\Delta_S^C,(x_t,b_t,K_t)\in\mathbb{C}_t} P_t((r_t,s_t)\in\Delta_D)(p_tRx_t) \\ +P_t((r_t,s_t)\in\Delta_L)b_t \\ +\delta(1-P_t((r_t,s_t)\in\Delta_D)-P_t((r_t,s_t)\in\Delta_L)-P_t((r_t,s_t)\in\Delta_S^C)E_tV_{t+1}(r_t,s_t) \end{array}
\end{aligned} \tag{6}$$

where the contract space $\mathbb{C}_t$ is given by:

$$\mathbb{C}_t=\{(x_t,b_t,K_t)\in[0,1]\times\mathbb{R}_+\times[0,\overline{K}]\}$$

such that

$$
\begin{aligned}
&P_t((r_t,s_t)\in\Delta_D)(p_tRx_t)+P_t((r_t,s_t)\in\Delta_L)b_t\\
&+\delta P_t((r_t,s_t)\in\mathbb{K}_{\Delta[0,1]}\backslash\Delta_D\cup\Delta_L\cup\Delta_S^C)E_tV_{t+1}(r_t,s_t)\\
&\qquad\geq K_t+\delta EV_{t+1}(r_{t-1},s_{t-1},r_t,s_t)
\end{aligned}
\tag{IC_t^{RL}}
$$

$$
\begin{aligned}
&P_t((r_t,s_t)\in\Delta_D)[(p_t-vq_t)R(1-x_t)-I]+P_t((r_t,s_t)\in\Delta_L)(L-b_t)\\
&\qquad\geq K_t
\end{aligned}
\tag{PC_t^{CF}}
$$

$$
\begin{aligned}
\text{if } (r_t,s_t) &\in \Delta_D, (p_t-vq_t)R(1-x_t)-I\geq L-b_t \qquad (IC_t^{CF})\\
\text{if } (r_t,s_t) &\in \Delta_L, (p_t-vq_t)R(1-x_t)-I < L-b_t
\end{aligned}
$$

The following proposition states that there is a unique optimal contract, and summarizes the optimal contracting terms and stopping rules. Figure illustrates the optimal stopping regions.

Proposition 2. *(1) There exists a unique Markov sequential equilibrium in the dynamic contracting game.*

(2) The contractual optima is given by the stopping rule:

$$
a_t^{CF}(r_t,s_t)=\begin{cases}
Dev & \textit{if } (r_t,s_t)\in\Delta_D\\
Liq & \textit{if } (r_t,s_t)\in\Delta_L\\
Abandon & \textit{if } (r_t,s_t)\in\Delta_S^C\\
Continue & \textit{otherwise}
\end{cases}
$$

where the optimal stopping thresholds are:

$$
\begin{aligned}
\Delta_D &= \left\{(r_t,s_t)\in\Delta_H|\ \left[p_tR-\frac{p_t}{(p_t-vq_t)}\left(I+\frac{1}{\lambda_t}\right)\right]\geq L-\frac{1}{\lambda_t}\right\}\\
\Delta_L &= \left\{(r_t,s_t)\in\Delta_H|\ \left[p_tR-\frac{p_t}{(p_t-vq_t)}\left(I+\frac{1}{\lambda_t}\right)\right]< L-\frac{1}{\lambda_t}\right\}\\
\Delta_S^C &= \left\{(r_t,s_t)\in\mathbb{K}_{\Delta_{[0,1]}}|\ L<\frac{2}{\lambda_t}\right\}
\end{aligned}
$$

The stopping time is:

$$
T:=\inf\{t|(r_t,s_t)\in\Delta_H\cup(r_t,s_t)\in\Delta_S^C\}
$$

The payment rules are:

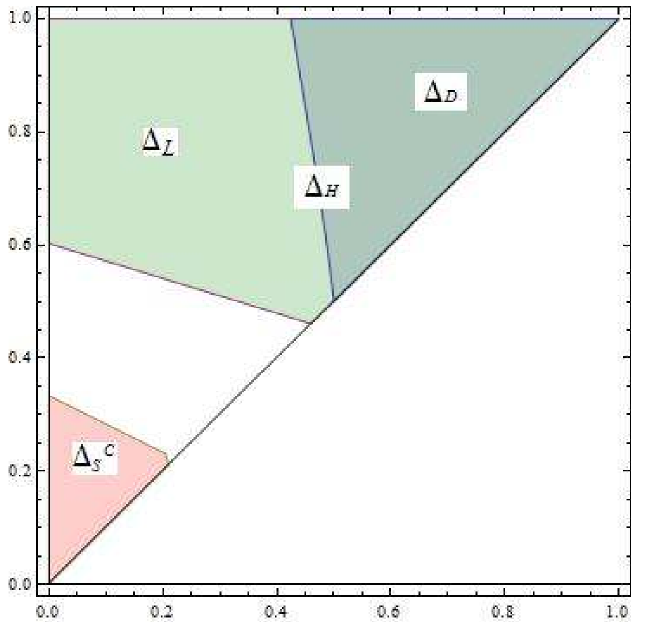

FIG. 4 Contractual Equilibrium

if $(r_t, s_t) \in \Delta_D$ $x_t = 1 - \frac{1}{R(p_t - vq_t)}\left(I + \frac{1}{\lambda_t}\right)$; $b_t \geq L - \frac{1}{\lambda_t}$	*if* $(r_t, s_t) \in \Delta_L$ $b_t = L - \frac{1}{\lambda_t}$ $x_t \geq 1 - \frac{1}{R(p_t - vq_t)}\left(I + \frac{1}{\lambda_t}\right)$

Proof. In Appendix A. ∎

Now we will turn to the funding pattern to answer the questions: a) Is it possible that the project will obtain full funding till the end, i.e., till the time the posterior $(r_t, s_t) \in \Delta_S^C$, b) If full funding is not available at all times, how does the funding flow evolve over time?

Define

$$\Delta_F \quad : \quad = \{(r_t, s_t) \in \mathbb{K}_{\Delta_{[0,1]}} \backslash \Delta_S^C | 0 < K_t < \overline{K} : \text{ the region of posterior beliefs where full funding is not available}\} \qquad (7)$$

There are three distinct cases:

Case 1: The project always receives full funding $\Delta_F = \phi$

Proposition 3 finds a sufficient condition on the initial prior for the project receiving full funding till the end.

Case 2: The project receives full funding whenever it is optimal to develop $\Delta_D \cap \Delta_F = \phi$

Case 3: The project does not receive full funding for all the beliefs for which it is optimal to develop $\Delta_D \cap \Delta_F \neq \phi$

Under this case, the ambiguity aversion of CF plays a role. Full funding region, i.e. the region where $(r_t, s_t) \in \Delta_D \cap \Delta_F^C$, increases with ambiguity aversion of CF. After full funding stops, i.e. for all $(r_t, s_t) \in \Delta_D \cap \Delta_F$, funding volume decreases over time. The intuition is same as in the two-period example. As CF becomes more ambiguity averse, the dynamic moral hazard problem is alleviated. CF, being ambiguity averse, becomes much more cautious and pessimistic after each low signal. So, following a low signal, CF has to be guaranteed a greater share of the final return in order to keep investing. Thus, the contractual terms sharing ambiguity also discipline RL and lower his dynamic expected value from cheating which, in turn, eases the funding constraint towards the beginning. Thus, if the project receives full funding in $\Delta_D \cap \Delta_F^C$, as v increases, this region increases. After the project stops receiving full funding, the investment flow is monotonically decreasing over time. This result is in contrast with the result in Bergemann and Hege (2005), where it is possible to have monotonically increasing investment pattern over time due to the severity of the dynamic agency problem.

PROPOSITION 3. *The project receives full funding under the following sufficient condition:*

$$\delta \geq \frac{2 - \lambda_0 L}{1 - \lambda_0[\bar{K} + \frac{L}{2}]} \tag{8}$$

If the project does not receive full funding till the end,

a) If $\delta \geq \frac{L}{L+1}$, $\Delta_D \cap \Delta_F = \phi$; *so full funding is available for all* $(r_t, s_t) \in \Delta_D$.

b) If $\delta < \frac{L}{L+1}$, $\Delta_D \cap \Delta_F \neq \phi$; *the project does not receive full funding for all* $(r_t, s_t) \in \Delta_D$. *In this case, as* v *increases, the project receives full funding for a longer time horizon, i.e.,* $\Delta_D \cap \Delta_F$ *shrinks.*

After full funding stops, investment volume monotonically decreases over time.

Proof. In Appendix A. ∎

Next section, we discuss the patent law if it is set by a non-strategic Policymaker and consequently the policy implications.

3. POLICY RECOMMENDATIONS

3.1. Patent Law

Assume that the patent law is set by the Policymaker (the patent-granting authority, or the regulatory agency), who is a risk and ambiguity neutral entity[16]. The Policymaker values the "open questions", or the "*Unknowable*" state more than the commercial firms do, hence is less ambiguity averse (for simplification, I assume ambiguity neutrality).

[16] Another benchmark we can possibly consider is a social planner's optima, where the social planner maximizes the aggregate welfare. Qualitatively the results remain the same whenever the planner puts a positive weight on the firm's utility.

Assume that the Policymaker cares only for the payoffs generated from the project[17]. Let us assume that the Policymaker sets the patent law to reflect his own desired outcome: the "Policymaker's Optima" , or, the "*Risk and Ambiguity Neutral Optima* (*RAN Optima*)" without considering the research alliances' response to it, so he is non-strategic.

After observing the signal at the end of each period, the Policymaker chooses whether to develop or to liquidate the project, or to continue experimenting further:

$$a_t^{RAN} \in \{Dev, Liq, Cont\}$$

The Policymaker's optimal stopping rule identifies the regions of posterior beliefs where it is optimal to stop experimenting and develop the project: Δ_H, and the region where it is optimal to stop experimenting and liquidate the project: Δ_S.

$$\begin{aligned} \Delta_H &= \{(r_t, s_t) \in \mathbb{K}_{\Delta[0,1]}|\ a_t^{RAN} = Dev\} \\ \Delta_S &= \{(r_t, s_t) \in \mathbb{K}_{\Delta[0,1]}|\ a_t^{RAN} = Liq\} \end{aligned}$$

Then, at the beginning of each period, the problem can be formulated recursively using the optimality equation or Bellman equation:

$$\begin{aligned} V_t^{RAN}(r_{t-1}, s_{t-1}) &= \max_{\Delta_H, \Delta_S, K_t^{RAN}} P_t((r_t, s_t) \in \Delta_H)(p_t R - I) + P_t((r_t, s_t) \in \Delta_S)L - K_t \\ &\quad + \delta[P_t((r_t, s_t) \in \mathbb{K}_{\Delta[0,1]} \backslash (\Delta_H \cup \Delta_S)]E_t V_{t+1}^{RAN}(r_t, s_t) \qquad (RAN\ ) \end{aligned}$$

PROPOSITION 4. *The RAN optima, or, the "Policymaker's Optima" is given by the stopping rule*

$$a_t^{RAN}(r_t, s_t) = \begin{cases} Dev & if\ (r_t, s_t) \in \Delta_H \\ Liq & if\ (r_t, s_t) \in \Delta_S \\ Continue & otherwise \end{cases}$$

where the optimal stopping thresholds are:

$$\begin{aligned} \Delta_H: &= \{(r_t, s_t)|\beta_{H1} r_t + \beta_{H2} s_t \geq \beta_{H3}\}; \\ \Delta_S: &= \{(r_t, s_t)|\beta_{S1} r_t + \beta_{S2} s_t < \beta_{S3}\} \end{aligned}$$

The stopping time is:

$$T_{RAN} = \inf\{t|(r_t, s_t) \in \Delta_H \cup (r_t, s_t) \in \Delta_S\}$$

[17] It might be argued that it is more natural to assume that the Policymaker would internalize the positive externalities the project might generate as well. Including these externalities will naturally make the contractual outcome diverge further from the risk and ambiguity neutral benchmark outcome, making our results even more robust.

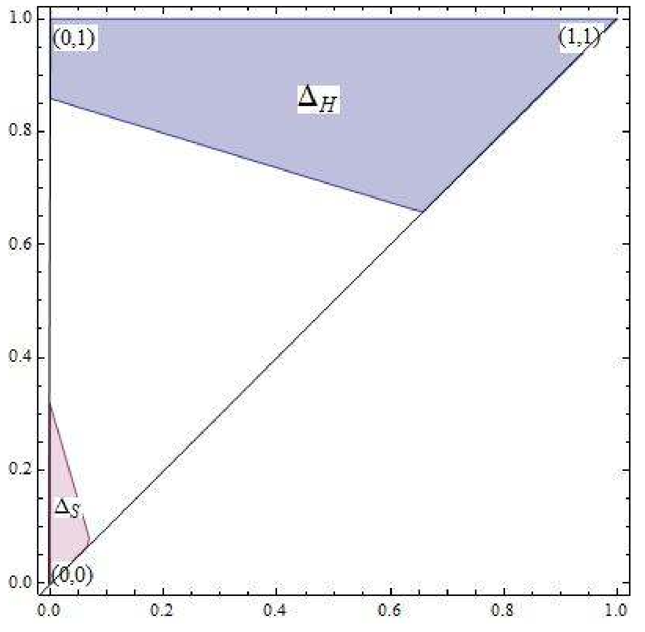

FIG. 5 Policymaker's Outcome

Also, the project receives full funding in every period it is continued.

$$K_t = \overline{K} \quad \forall t \leq T_{RAN}$$

Proof. In Appendix A. ∎

Thus, the Policymaker's value from this innovation project is:

$$V_0^S = E_0 \left[\sum_{t=1}^{T_S} \delta^{t-1} \left(P_t((r_t, s_t) \in \Delta_H)(p_t R - I) + P_t((r_t, s_t) \in \Delta_S) L - \overline{K} \right) \right] \quad (9)$$

The patent law threshold for a parametric example is depicted in the Figure 5.

3.2. Policy Recommendations

Now that we have analyzed the contractual equilibrium within the research alliance, and the Policymaker's desired outcome, we can compare them and evaluate the alliance as a mode of organizing research. Notice that in the contractual scenario, there are three possible sources of deviation from the RAN outcome. Firstly, the static and dynamic moral hazard can potentially distort the incentives and make it harder for the project to obtain funding at every period, thereby creating a divergence from the optima the Policymaker intends to implement. Also, the presence of ambiguity and $CF's$ ambiguity aversion creates a divergence in preferences among the strategic alliance and the Policymaker, thus contributing to the difference from the RAN optima. Lastly, the short term contracting and lack of commitment can result in the contractual outcome being different that the RAN optima. Let us first examine how these possible sources of inefficiencies interact with each other and result in a divergence in the desired outcome and the contractual outcome.

The Policymaker's value from the project carried out by the strategic partnership is given by:

$$V_0^{SC} = E_0 \left[\begin{array}{c} \sum_{t=1}^{T} \delta^{t-1} [P_t((r_t, s_t) \in \Delta_D)(p_t R - I) + P_t((r_t, s_t) \in \Delta_L) L \\ -(1 - P_t((r_t, s_t) \in \Delta_F))\overline{K} - P_t((r_t, s_t) \in \Delta_F) K_t] \end{array} \right] \tag{10}$$

Comparing 9 and 10, we see that the contractual outcome diverges from the Policymaker's outcome in three ways:

(a) Patent Troll: If the posterior belief $(r_t, s_t) \in \Delta_L \subset \Delta_H$, the risk and ambiguity neutral Policymaker finds it optimal to develop the product, but because of $CF's$ ambiguity aversion, the strategic partnership liquidates the product even after being granted patent. So, every time the posterior lies in this region, there is a loss of value $p_t R - I - L > 0$ to the Policymaker. This loss is attributed to the difference in ambiguity attitude of the Policymaker and CF.

(b) Less experimentation: The Policymaker optimally stops experimentation and abandons the project as soon as the posterior belief enters Δ_S, while the research alliance abandons it when the posterior lies in Δ_S^C, where $\Delta_S \subset \Delta_S^C$. So the research alliance abandons the project for a larger range of posterior beliefs, compared to the Policymaker. This result is due to the short termism, lack of commitment power of the research alliance, and the moral hazard problem.

(c) Partial Funding: The Policymaker optimally invests the maximal funding in the project till the end, whereas the research partnership, if the prior belief is not too high (if 8 is not satisfied), does not receive full funding till the end. The lower investment flow is driven by the static and dynamic moral hazard problem, which makes the incentive constraints harder to satisfy. However, as we have noted in Proposition 3, dynamic moral hazard problem is alleviated as v goes up, causing the project to receive maximal funding for a longer time horizon.

The next proposition summarizes how the equilibrium contractual outcome diverges from the Policymaker's optimal outcome.

PROPOSITION 5. *Compared to the Policymaker's optima, the equilibrium contracts governing the research alliances result in (a) liquidation of the project even after being patented, (b) less experimentation, and (c) lower investment flow.*

Proof. In Appendix A. ∎

The following figure illustrates the difference between the two outcomes.

Given that the contracts governing the strategic partnerships fail to implement the Policymaker's optima, next we examine if the Policymaker can restructure the patent

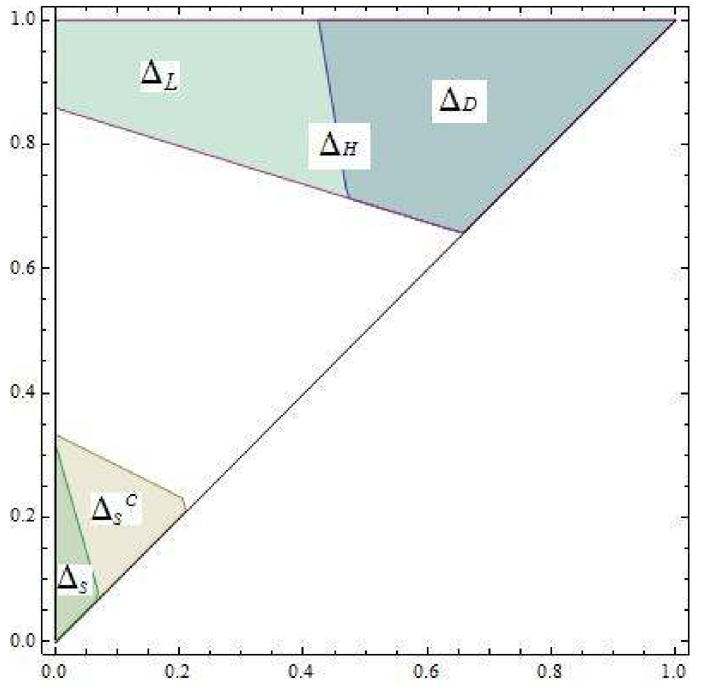

FIG. 6 Comparison Between Contractual Outcome and Policymaker's Outcome

law in order to implement its desired optima. Specifically, if the patent law is designed to internalize the possible response from the research alliances, is it possible to alleviate the three sources of inefficiency discussed above? Analyzing the effects of changing the patent law, we find that if the patent law is made stricter, i.e., Δ_H is set at a higher level, it will shrink Δ_L, so it is less likely that the project will be liquidated after being granted patent. However, this would lower the incentive to experiment as well, because $P_t((r_t, s_t) \in \Delta_H)$ decreases, causing the research alliance to abandon the project even earlier (for a larger range of posteriors) than before. In fact, setting $\Delta_H = \Delta_D$ eliminates the possibility of patent troll, but increases the range of posteriors for which the project is abandoned forever; i. e. , Δ_S^C expands.

On the other hand, if the patent policy is relaxed, that boosts the incentive to invest in the project, increasing $P_t((r_t, s_t) \in \Delta_H)$ at every period, and results in longer experimentation and higher level of investment. However, it also results in an expansion of Δ_L, so patent troll problem becomes more severe. Thus, changing the patent law can never fully implement the Policymaker's optima and eliminate all three sources of efficiency. If initially Δ_L is large, i.e., patent troll is a severe problem to start off with, then making the patent law more stringent benefits the Policymaker more, whereas if the inefficient stopping proves to be a more severe concern, then relaxing the patent policy would be beneficial. So, restructuring the patent law can not implement the Policymaker's outcome.

3.3. Generalizations

3.3.1. General signal structure

In this model, we have used the simplifying assumption of linearity in the signal structure. This resulted in the Policymaker's optima characterized by full funding at all times.

With a more general signal structure satisfying only the Maximum Likelihood Ration Property, instead of full funding, the optimal outcome will be characterized by a partial investment flow that decreases over time for the Policymaker as well as the strategic partnership. The regions Δ_D, Δ_L, and Δ_S^C can be characterized likewise. The main results qualitatively stays the same.

A more general signal structure instead of the binary signal will change the results significantly. Indeed, in some real life contexts, the information flow that arrives at each period of experimentation can not be encoded into a simple binary signal. Assuming a continuous signal structure will generalize the model and consequently change the optimal contract structure.

3.3.2. No Limited Liability of RL

In the present model, the research lab is assumed to be liquidity constrained, thus always requires a non-negative payment in each period. However, in many real life scenario, the research based firms, though smaller in comparison to the commercial giants, can afford to put forth some investment, in the form of collateral , in order to continue experimentation even after clearing the patent thresholds. Under this assumption, experimentation may continue even after clearing the patenting threshold and there is a possible non-monotonicity of experimentation; the patent troll region shrinks, and the alliance experiments longer.

3.3.3. Long Term Contracts

In some situations, firms can attain commitment power through brand reputations, press releases and a variety of other ways. If the contracting parties can commit to long term relations, the participation constraint of CF will not have to be met in every period, so intertemporal transfer of payments will be possible. This relaxes the funding condition at every period and results in longer experimentation. In this case, experimentation may continue even after being granted a patent and the patent troll region shrinks.

3.3.4. Partially Observable Signal

In many scenario, the informative signal is not publicly revealed. Sometimes, the financing firm hires experts to evaluate the reports given by the research firm, whose evaluation criteria varies from the research firm. It is also possible that the results from the experimentation can be mis-reported. In these cases, the assumption that the signal at each period is publicly observed breaks down. One can explore the existence of a strategy-proof contract under such partial observability and possible mis-reporting of the signals.

4. DISCUSSION

In the innovation intensive industries, we observe that research partnership is increasingly becoming an important mode of organizing research. The results from this paper suggest that the policy making organizations should recognize the fact and be aware of how the innovation activity conducted in the research alliances is affected by the patent policy. Using the predictions from the theoretical model, we observe that relaxing the patent criteria is likely to result in longer experimentation, but at the same time the possibility of patent troll like cases increases; whereas if the patent law is made more stringent then the patented projects are more likely to be developed, but the research alliances stop experimenting inefficiently early. This result suggests that studying the present state of the industry, the patent authority should decide on the patent criterion.

Also, comparing the optimal contractual outcome and the Policymaker's optima, we can see that it is never possible to implement the Policymaker's optima. As the contextual ambiguity associated with the project increases, the divergence between the contractual outcome and the desired outcome increases. This suggests that the projects with high level of ambiguity can not be satisfactorily organized by research partnerships. Indeed, there can be projects, which the Policymaker deems profitable enough to invest in, that can be never funded in a research partnership. In innovative industries, the concern about important innovations not being carried out has long been voiced (Clayton Christensen, ITExpo, 2011). The industry's Internal Rate of Return Criterion and lack of foresight are often blamed as the root causes for not investing in innovative technologies.

This suggests a potential role of a regulatory body or the "State" as an entrepreneur. State intervention in innovation in the form of funding programs for smaller research oriented firms can support innovation organized in research firms. State programs for Small and Medium Enterprises (SMEs) and New Biotechnology Firms (NBFs) like Small Business Innovation Research (SBIR), 1982, Small Business Technology Transfer (STTR), 1992 have been able to fund numerous ventures by smaller research firms and touted as success(SBIR/STTR Impact Report, 2012). In the US, 57% of "basic research" is supported through Federal funding (NSF report, 2008). Programs such as these, providing funds to the research oriented smaller firms, lead to the development of the projects not otherwise funded (Mazzucato (2013)).

Another mode of organizing innovation when the research alliances can not efficiently carry it out is direct state initiative. There are several examples where State as an entrepreneur has participated in innovation and led to successful development of projects. In UK, Medical Research Council (MRC), funded by the Department for Business, Innovation and Skills (BIS) has been leading the Pharmaceutical innovation and was behind the development of monoclonal antibodies, widely used in Pharmaceutical industry since then. In the US, National Institute of Health (NIH) has been key funding source for research in Biotechnology, spending $30.9 bn in 2012 alone. Another example of State's entrepreneurial venture is National Nanotechnology Initiative (NNI), which, funded in 2000, strives to engage in cutting edge research in Nanotechnology. According to the famous adage by Polanyi (1944):

> "The road to the free market was opened and kept open by an enormous increase in continuous, centrally organized and controlled interventionism."

5. SUMMARY AND CONCLUSION

Research alliances are responsible for a major share of innovation activity in the research-intensive industries. The innovation processes they undertake is often characterized by ambiguity rather than risk. Given the prevalence of these research alliances in these sectors, it is important to examine the optimal research outcome that is generated in these R&D partnerships, understand the strategic incentives of the contracting parties and how these interact to shape the optimal choices, and to evaluate the research alliance as a mode of organizing research in the ambiguous environment. This paper provides a theoretical framework to analyze these partnerships and compare it to the optimal outcome that a risk and ambiguity neutral Policymaker wants to implement.

Apart from the different extensions and robustness issues mentioned in the previous section, this study can open up the path of further research on strategic partnerships. It will be interesting to study multi-lateral strategic partnerships in the innovation-based industries as networks and examine the optimal network structure that emerges under ambiguity with different parametric assumptions. Also, analyzing different patent policies in this context under ambiguity constitutes another interesting direction for future research.

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APPENDIX: FOR ONLINE PUBLICATION

Appendix A: Proofs

Proof of Proposition 1. Using a few lemmata we get to the main result of the two period example, captured in Proposition 1.

Equilibrium delay occurs when the incentive constraint in $t = 1$ can not be satisfied, but in $t = 2$ can be. The range of posterior beliefs for which this happens is given in 5:

$$\max\{\mu_2 R p_2, \mu_2 b_2\} \geq \bar{K} > \max\{\mu_1 R p_1, \mu_1 b_1\} - DAC_2$$
$$\Leftrightarrow \frac{1 + \lambda_1 p_1 \left(\frac{1}{(p_1 - vq_1)\lambda_1} + \frac{I}{p_1 - vq_1}\right) - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2 \left(\frac{1}{(p_2 - vq_2)\lambda_2} - \frac{I}{p_2 - vq_2}\right)\right)}{\lambda_1 p_1 - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2\right)}$$
$$> R \geq \frac{1 + \lambda_2 p_2 \left(\frac{1}{(p_2 - vq_2)\lambda_2} + \frac{I}{p_2 - vq_2}\right)}{\lambda_2 p_2} \qquad (11)$$

For the sake of brevity, define:

$$T_1 = \frac{1 + \lambda_1 p_1 \left(\frac{1}{(p_1 - vq_1)\lambda_1} + \frac{I}{p_1 - vq_1}\right) - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2 \left(\frac{1}{(p_2 - vq_2)\lambda_2} - \frac{I}{p_2 - vq_2}\right)\right)}{\lambda_1 p_1 - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2\right)}$$
$$T_2 = \frac{1 + \lambda_2 p_2 \left(\frac{1}{(p_2 - vq_2)\lambda_2} + \frac{I}{p_2 - vq_2}\right)}{\lambda_2 p_2}$$

Define the range of posterior beliefs for which delay happens as:

$$\Delta_{Delay}(v) = \{(r_1, s_1) | T_1 > R > T_2\} \qquad (12)$$

If $T_1 < T_2$, delay can never happen. So the first step is identify the region of posterior belief for which delay happens, when both the parties are ambiguity neutral, i.e. $v = 0$. Then we can show that $(T_1 - T_2)$ decreases with v.

LEMMA 1. *For $v = 0$, i. e. , if the principal is ambiguity neutral, then*

$$T_1 > T_2$$

So, in equilibrium delay is possible.

Proof. If $v = 0$,

$$T_1 = \frac{2 + \lambda_1 I - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 I\right)}{\lambda_1 p_1 - \delta\left(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2\right)}$$
$$T_2 = \frac{2 + \lambda_2 I}{\lambda_2 p_2}$$

Hence,

$$\begin{aligned} T_1 - T_2 &= \frac{[I\lambda_2 p_2 \lambda_1 \delta + \lambda_1 p_1 (\delta - (1-\delta)\lambda_2 I)]}{(\lambda_1 p_1 - \delta(\lambda_1 p_1 - (1-\mu_1)\lambda_2 p_2))(\lambda_2 p_2)} \\ &> 0 \\ &\Leftrightarrow \Delta_{Delay}(v=0) \neq \phi \end{aligned}$$

∎

Lemma 2. *If the discount factor is not too high, $\delta \leq \overline{\delta} < 1$, for all $v \in [0,1]$, as v increases, $T_1 - T_2$ falls, where $\overline{\delta}$ is given by:*

$$\overline{\delta} = 1 - \left(\frac{p_2 - vq_2}{p_1 - vq_1}\right)^2 \frac{q_1}{q_2} \left(\frac{\frac{1}{\lambda_1} + I}{\frac{1}{\lambda_2} + I}\right)$$

The proof follows directly from taking derivatives. Next, we prove the existence of a threshold value of $v = \widetilde{v}$ for which delay does not happen.

Lemma 3. *There exists $\widetilde{v} \in (0,1)$ for which $T_1 = T_2$.*

Proof.

$$T_1 - T_2 = \frac{1}{(p_2 - vq_2)(p_1 - vq_1)} \left[\begin{array}{c} (p_2 - vq_2)\left(\frac{1}{\lambda_1} + I\right) - (1-\delta)(p_1 - vq_1)\left(\frac{1}{\lambda_2} + I\right) \\ -(p_1 - vq_1)(p_2 - vq_2)\left\{\lambda_1 p_1 (1-\delta) + \lambda_2 p_2 [\delta \lambda_1 p_1 + \delta(1-\mu_1) - 1]\right\} \end{array} \right]$$

For $v = 1$,

$$T_1 - T_2|_{v=1} = \frac{1}{(p_2 - q_2)(p_1 - q_1)} \left[\begin{array}{c} (p_2 - q_2)\left(\frac{1}{\lambda_1} + I\right) - (1-\delta)(p_1 - q_1)\left(\frac{1}{\lambda_2} + I\right) \\ -(p_1 - q_1)(p_2 - q_2)\left\{\lambda_1 p_1 (1-\delta) + \lambda_2 p_2 [\delta \lambda_1 p_1 + \delta(1-\mu_1) - 1]\right\} \end{array} \right]$$

Now,

$$(p_2 - q_2)\left(\frac{1}{\lambda_1} + I\right) - (1-\delta)(p_1 - q_1)\left(\frac{1}{\lambda_2} + I\right) \leq 0$$

$$\Leftrightarrow \delta \leq 1 - \frac{p_2 - q_2}{p_1 - q_1} \frac{\frac{1}{\lambda_1} + I}{\frac{1}{\lambda_2} + I} \tag{13}$$

And

$$\begin{aligned} &\lambda_1 p_1 (1-\delta) + \lambda_2 p_2 [\delta \lambda_1 p_1 + \delta(1-\mu_1) - 1] \\ =\ &(1-\delta)(\lambda_1 p_1 - \lambda_2 p_2) + \lambda_2 p_2 \delta [\lambda_1 p_1 - \mu_1] > 0 \end{aligned}$$

Since

$$1 - \frac{p_2 - q_2}{p_1 - q_1} \frac{\frac{1}{\lambda_1} + I}{\frac{1}{\lambda_2} + I} \quad > \quad \overline{\delta} = 1 - \left(\frac{p_2 - q_2}{p_1 - q_1}\right)^2 \frac{q_1}{q_2} \left(\frac{\frac{1}{\lambda_1} + I}{\frac{1}{\lambda_2} + I}\right),$$
$$\Rightarrow \quad \forall \delta \leq \overline{\delta}, T_1 - T_2|_{v=1} < 0$$

So, $T_1 - T_2$ is continuous in v and it decreases as v increases. Also, $T_1 - T_2|_{v=0} > 0$ and $T_1 - T_2|_{v=1} < 0$, hence there must exist a $\widetilde{v} \in (0, 1)$, for which $T_1 = T_2$.

Hence,

$$\Delta_{Delay}(v) = \phi \text{ for all } v > \widetilde{v}$$

▮

Proof of Proposition 2. Since CF stops experimenting the first time the posterior crosses the patenting threshold, RL only chooses the contract to offer depending on whether developing the project after being patented is more beneficial than liquidating. Thus, whenever $RL's$ expected payoff if CF develops the product: $\mu_t p_t \left[R - \frac{(I + \frac{1}{\lambda_t})}{p_t - vq_t}\right]$ is greater than the expected payoff if CF liquidates: $L - \frac{1}{\lambda_t}$, he chooses

$$x_t = 1 - \frac{\left(I + \frac{1}{\lambda_t}\right)}{R(p_t - vq_t)}, b_t \geq L - \frac{1}{\lambda_t} \tag{14}$$

and the reverse otherwise. This gives us Δ_D, Δ_L.

The project is abandoned when no contract satisfying both the incentive constraint for RL and the participation constraint for CF can be offered. Combining both the constraints, it is most difficult to hold if $(r_t, s_t) \in \Delta_L$:

$$L - \frac{1}{\lambda_t} \geq \frac{1}{\lambda_t} \tag{15}$$

So, the project is abandoned if

$$(r_t, s_t) \in \Delta_S^C = \left\{(r_t, s_t) | L < \frac{2}{\lambda_t}\right\}$$

The value function 6 can be rewritten as:

$$\begin{aligned} &V_t(r_{t-1}, s_{t-1}) \\ &= \begin{array}{c} \max_{\Delta_D, \Delta_L, \Delta_S^C} P_t((r_t, s_t) \in \Delta_D) p_t R \left(1 - \frac{(I + \frac{1}{\lambda_t})}{R(p_t - vq_t)}\right) \\ + P_t((r_t, s_t) \in \Delta_L) \left(L - \frac{1}{\lambda_t}\right) \\ + \delta(1 - P_t((r_t, s_t) \in \Delta_D) - P_t((r_t, s_t) \in \Delta_L) - P_t((r_t, s_t) \in \Delta_S^C) E_t V_{t+1}(r_t, s_t) \end{array} \end{aligned} \tag{16}$$

We can define the operator $\Upsilon : \mathbb{R} \to \mathbb{R}$ as:

$$\Upsilon(V) = \begin{array}{c} \max_{\Delta_D,\Delta_L,\Delta_S^C} P((r^{'},s^{'}) \in \Delta_D) pR\left(1 - \frac{(I+\frac{1}{\lambda})}{R(p-vq)}\right) \\ +P((r^{'},s^{'}) \in \Delta_L)\left(L - \frac{1}{\lambda}\right) \\ +\delta(1 - P((r^{'},s^{'}) \in \Delta_D) - P((r^{'},s^{'}) \in \Delta_L) - P((r^{'},s^{'}) \in \Delta_S^C) EV(r^{'},s^{'}) \end{array}$$

Properties of Υ **:**

Monotonicity:

As $V(r,s) \le V^1(r,s)$, $\Upsilon(V) \le \Upsilon(V^1)$ $\forall (r,s) \in \mathbb{K}_{\Delta_{[0,1]}}$.

Discounting:

The discount factor $\delta \in (0,1)$ ensures that

$$[\Upsilon(V+a)](r,s) \le \Upsilon(V)(r,s) + \delta a$$

for all $V, a \ge 0, (r,s) \in \mathbb{K}_{\Delta_{[0,1]}}$. ∎

Hence Υ satisfies Blackwell's sufficiency conditions (Theorem 3.3 in Stokey (1989)), so it is a contraction. Then, by directly using the Contraction Mapping Theorem (Theorem 3.2 in Stokey (1989)), we show that T has exactly one fixed point V that solves the contracting problem. ∎

Proof of Proposition 3.

To examine the funding flow, first let us look at the incentive constraint RL faces at any t.

If $(r_t, s_t) \in \Delta_D$, the dynamic incentive constraint is:

$$\mu_t p_t R x_t + (1-\mu_t)\delta EV_{t+1}(r_t,s_t) \ge K_t + \delta EV_{t+1}(r_{t-1},s_{t-1},r_t,s_t)$$

Substituting for the optimal share x_t from 2, rewrite it as:

$$\mu_t p_t \left(R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right) + (1-\mu_t)\delta EV_{t+1}(r_t,s_t) \\ \ge K_t + \delta EV_{t+1}(r_{t-1},s_{t-1},r_t,s_t)$$

Now, the dynamic expected payoff to be collected by RL in future periods following a diversion can be expressed as:

$$EV_{t+1}(r_{t-1},s_{t-1},r_t,s_t) = \frac{\lambda_{t-1}p_{t-1}}{\lambda_t p_t} EV_{t+1}(r_t,s_t) \tag{17}$$

Using this, the dynamic IC IC_t^{RL} can be rewritten as:

$$\mu_t p_t \left(R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right) - K_t \ge \delta\left[\frac{\lambda_{t-1}p_{t-1}}{\lambda_t p_t} - (1-\mu_t)\right] EV_{t+1}(r_t,s_t) \tag{18}$$

where the RHS captures the dynamic agency cost.

Similarly, if $(r_t, s_t) \in \Delta_L$, the dynamic incentive constraint can be rewritten as:

$$\mu_t\left(L - \frac{1}{\lambda_t}\right) - K_t \geq \delta\left[\frac{\lambda_{t-1}}{\lambda_t} - (1-\mu_t)\right] EV_{t+1}(r_t, s_t) \tag{19}$$

and it does not depend on $CF's$ ambiguity aversion.

The first lemma finds the sufficient conditions under which the project receives full funding till the end. ∎

Lemma 4. *Sufficient condition for the project to obtain full funding till the end is:*

$$\lambda_0 \geq \frac{2-\delta}{L(1-\frac{\delta}{2}) - \delta\overline{\overline{K}}}$$

Proof. Let us look at the last period T, after which the project is abandoned forever. At T^{th} period, the incentive constraint binds:

$$\mu_T\left[L - \frac{1}{\lambda_T}\right] = K_T \Leftrightarrow \lambda_T = \frac{2}{L}$$

So,

$$EV_T(r_{T-1}, s_{T-1}) = K_T$$

At the penultimate period, the dynamic IC is:

$$\mu_{T-1}(L - \frac{1}{\lambda_{T-1}}) - K_{T-1} \geq \delta\left[\frac{\lambda_{T-1}}{\lambda_T} - (1-\mu_{T-1})\right] EV_T(r_{T-1}, s_{T-1})$$
$$\Longleftrightarrow \mu_{T-1}(L - \frac{1}{\lambda_{T-1}}) - K_{T-1} \geq \delta\left[\frac{\lambda_{T-1}}{\lambda_T} - (1-\mu_{T-1})\right] K_T$$

This incentive constraint is most difficult to satisfy if $K_T = K_{T-1} = \overline{K}$. Thus, the project receives full funding till the end if:

$$\lambda_{T-1}L - 2 \geq \delta\frac{\lambda_{T-1}}{\lambda_T} - 1 + \lambda_{T-1}\overline{K}$$
$$\Leftrightarrow \lambda_{T-1} \geq \frac{2-\delta}{L(1-\frac{\delta}{2}) - \delta\overline{\overline{K}}} \quad (\text{using } \lambda_T = \frac{2}{L})$$

The sufficient condition becomes:

$$\lambda_0 \geq \frac{2-\delta}{L(1-\frac{\delta}{2}) - \delta\overline{\overline{K}}} \tag{20}$$
$$\Leftrightarrow \delta \geq \frac{2-\lambda_0 L}{1-\lambda_0[\overline{K} + L/2]} \tag{21}$$

Now let us analyze Case 2 and Case 3. If the project does not receive full funding till the end, we want to characterize the switching point, i.e. the posterior beliefs at which the investment flow switches from full funding to partial funding. To characterize the equilibrium switching point, we derive the difference equation for $CF's$ funding decision, provided the IC_t^{RL} is binding under restricted funding.

Case 2: $\Delta_F \cap \Delta_D = \phi$: At the switching point. after being patented, the project is liquidated.

Case 3: $\Delta_F \cap \Delta_D \neq \phi$: At the switching point, after being granted a patent, the project is developed till the end.

First, let us focus on Case 2.

LEMMA 5. *If $\Delta_F \cap \Delta_D = \phi$, then the switching point can be given as a quadratic equation in (r_t, s_t) :*

$$\Phi_L(r_t, s_t) = \gamma_{L1} r_t^2 + \gamma_{L2} s_t^2 + \gamma_{L3} r_t s_t + \gamma_{L1} = 0 \tag{22}$$

∎

LEMMA 6. *and*

$$\Delta_F = \{(r_t, s_t) | \Phi_L(r_t, s_t) < 0\}$$

Proof. The expected value of RL along the equilibrium path can be represented as:

$$EV_t(r_{t-1}, s_{t-1}) = \mu_t(L - \frac{1}{\lambda_t}) + \delta(1-\mu_t) EV_{t+1}(r_t, s_t) \tag{23}$$

Now, if the project receives restricted funding at time t, IC_t^{RL} binds on the equilibrium path, so:

$$\mu_t(L - \frac{1}{\lambda_t}) - K_t = \delta \left[\frac{\lambda_t}{\lambda_{t+1}} - (1-\mu_t) \right] EV_{t+1}(r_t, s_t)$$

Using the Bayesian updating:

$$\begin{aligned} \lambda_{t+1} &= \frac{(\lambda_G - \lambda_U) r_{t-1}(1 - K_t \lambda_t) + 1 - \lambda_t K_t - (1 - s_{t-1})(1 - \lambda_B K_t)(\lambda_U - \lambda_B)}{1 - \lambda_t K_t} \\ &= \frac{A_t - B_t K_t}{1 - \mu_t} = \frac{\lambda_t - B_t K_t}{1 - \lambda_t K_t} \end{aligned}$$

where A_t and B_t are given by:

$$\begin{aligned} A_t &= (\lambda_G - \lambda_U) r_{t-1} - (1 - s_{t-1})(\lambda_U - \lambda_B) + 1 = \lambda_t \\ B_t &= (\lambda_G - \lambda_U) r_{t-1} \lambda_t + (\lambda_t - \lambda_B)(1 - s_{t-1})(\lambda_U - \lambda_B) \end{aligned}$$

Simplifying, we get

$$B_t = \lambda_t^2 + (\lambda_t - \lambda_B)(1 - s_{t-1})(\lambda_U - \lambda_B) > \lambda_t^2$$

$$
\begin{aligned}
EV_{t+1}(r_t, s_t) &= \frac{K_t[\lambda_t L - 2]}{\delta\left[\frac{\lambda_t}{\lambda_{t+1}} - 1 + \lambda_t K_t\right]} && (24)\\
&= \frac{[\lambda_t L - 2][\lambda_t - B_t K_t]}{\delta B_t[1 - \lambda_t K_t]} && (25)\\
&= h_L(K_t)
\end{aligned}
$$

Now,

$$
\begin{aligned}
& h_L(K_t) \\
= \; & \frac{[\lambda_t L - 2]}{\delta B_t\left[\frac{1-\lambda_t K_t}{\lambda_t - B_t K_t}\right]}; \forall K_t \neq \frac{\lambda_t}{B_t}
\end{aligned}
$$

And,

$$
\begin{aligned}
& \frac{\partial}{\partial K_t}\left[\frac{1 - \lambda_t K_t}{\lambda_t - B_t K_t}\right] \\
= \; & \left[\frac{B_t - \lambda_t^2}{(\lambda_t - B_t K_t)^2}\right] > 0; \forall K_t \neq \frac{\lambda_t}{B_t}
\end{aligned}
$$

So,

$$
\frac{\partial h_L}{\partial K_t} > 0, \text{ for } K_t \neq \frac{\lambda_t}{B_t} \qquad (26)
$$

Substituting 24 into 23, we obtain:

$$
\begin{aligned}
EV_t(r_{t-1}, s_{t-1}) &= \mu_t(L - \frac{1}{\lambda_t}) + \delta(1 - \mu_t)h_L(K_t) \\
&= \mu_t(L - \frac{1}{\lambda_t}) + \frac{[\lambda_t L - 2][\lambda_t - B_t K_t]}{B_t}
\end{aligned}
$$

Moving it one period forward, an alternative expression for $EV_{t+1}(r_t, s_t)$ is found:

$$
\begin{aligned}
EV_{t+1}(r_t, s_t) &= \mu_{t+1}(L - \frac{1}{\lambda_{t+1}}) + \frac{[\lambda_{t+1} L - 2][\lambda_{t+1} - B_{t+1} K_{t+1}]}{B_{t+1}} && (27)\\
&= K_{t+1}(\lambda_{t+1}(K_t)L - 1) + \frac{[\lambda_{t+1}(K_t)L - 2][\lambda_{t+1}(K_t) - B_{t+1}(K_t)K_{t+1}]}{B_{t+1}(K_t)} && (28)\\
&= g_L(K_t, K_{t+1})
\end{aligned}
$$

where

$$
\frac{\partial g_L}{\partial K_t} \leq 0, \frac{\partial g_L}{\partial K_{t+1}} > 0
$$

Then, the difference equation with restricted funding is obtained by equating 23 and 27:

$$
g_L(K_t, K_{t+1}) = h_L(K_t) \qquad (29)
$$

By Implicit function theorem,

$$\begin{aligned}\frac{dK_{t+1}}{dK_t} &= -\frac{\frac{\partial g_L}{\partial K_t}-\frac{\partial h_L}{\partial K_t}}{\frac{\partial g_L}{\partial K_{t+1}}}\\ &\geq 0\end{aligned}$$

Thus, the difference equation 29 expresses K_{t+1} as an increasing function of K_t. This ensures the existence of a fixed point of the equation 29 at the full funding level, denoted by:

$$\overline{K}\,(\lambda_{t+1}L-1)+\frac{[\lambda_{t+1}L-2][\lambda_{t+1}-B_{t+1}\overline{K}]}{B_{t+1}}=\frac{[\lambda_t L-2][\lambda_t-B_t\overline{K}]}{\delta B_t[1-\lambda_t\overline{K}]}$$

which can be succinctly rewritten as the quadratic equation:

$$\Phi_L(r_{t-1},s_{t-1})=\gamma_{L1}r_{t-1}^2+\gamma_{L2}s_{t-1}^2+\gamma_{L3}r_{t-1}s_{t-1}+\gamma_{L1}=0 \tag{30}$$

This denotes the switching point. Δ_F is the area below the switching point:

$$\Delta_F=\{(r_t,s_t)\in\mathbb{K}_{\Delta_{[0,1]}}\backslash\Delta_S^C|\Phi_L(r_t,s_t)\leq 0\}$$

∎

Next lemma establishes that the switching point given by 22 indeed lies above the stopping threshold, i.e., $\Delta_F\subset\mathbb{K}_{\Delta_{[0,1]}}\backslash\Delta_S^C$, and also finds the range of parameters for which the project is liquidated at the switching point, i.e. $\Delta_F\cap\Delta_D=\phi$.

LEMMA 7. *The switching point locus always lies above the optimal stopping threshold, i.e.* $\Delta_F\subset\mathbb{K}_{\Delta_{[0,1]}}\backslash\Delta_S^C$

If $\delta\geq\frac{L}{L+1}$, *the project receives full funding for all beliefs* $(r_t,s_t)\in\Delta_D$, *i.e.* $\Delta_F\cap\Delta_D=\phi$.

Proof. We show that at the last period, the posterior belief is such that the project will not receive full funding.

At $t=T$, $L=\frac{2}{\lambda_T}$. Plugging this in 22, it is shown that, if the sufficiency condition does not hold,

$$\begin{aligned}\Phi_L(r_T,s_T) &= \overline{K}\,(\lambda_T L-1)+\frac{[\lambda_T L-2][\lambda_T-B_T\overline{K}]}{B_T}-\frac{[\lambda_{T-1}L-2][\lambda_{T-1}-B_{T-1}\overline{K}]}{\delta B_{T-1}[1-\lambda_{T-1}\overline{K}]}\\ &= \overline{K}-\frac{[\lambda_{T-1}L-2][\lambda_{T-1}-B_{T-1}\overline{K}]}{\delta B_{T-1}[1-\lambda_{T-1}\overline{K}]}\\ &< B_{T-1}[\delta\left(1-\frac{2\overline{K}}{L}\right)]-\lambda_{T-1}(\lambda_{T-1}L-2)\\ &< 0 \quad (\text{since } L<2\overline{K})\end{aligned}$$

Similarly, whenever it is optimal to liquidate after being granted patent rights, i.e.

whenever

$$
\begin{aligned}
& p_t \left[R - \frac{1}{p_t - vq_t} \left(I + \frac{1}{\lambda_t} \right) \right] < L - \frac{1}{\lambda_t} \\
\Leftrightarrow & \lambda_t p_t \left[R - \frac{1}{p_t - vq_t} \left(I + \frac{1}{\lambda_t} \right) \right] < (\lambda_t L - 1) \\
\Leftrightarrow & \lambda_t < \frac{vq_t}{(p_t R - L)(p_t - vq_t) - p_t} \equiv \tilde{\lambda} \qquad (31)
\end{aligned}
$$

we can show that under some parametric ranges full funding is not available for the project. If 31 is satisfied, then

$$
\begin{aligned}
& \Phi_L(r_t, s_t) < 0 \\
\Leftrightarrow & B_{t+1} B_t \delta \overline{K}(1 - \lambda_t \overline{K})(\lambda_{t+1} L - 1) \\
& + \delta B_t (1 - \lambda_t \overline{K})(\lambda_{t+1} L - 2)(\lambda_{t+1} - B_{t+1} \overline{K}) \\
& - B_{t+1}(\lambda_t L - 2)(\lambda_t - B_t \overline{K}) < 0
\end{aligned}
$$

Now,

$$
\begin{aligned}
& LHS \\
< \; & \lambda_{t+1} \lambda_t \delta \overline{K}(1 - \lambda_t \overline{K})(\lambda_{t+1} L - 1) \\
& + \delta \lambda_t (1 - \lambda_t \overline{K})(\lambda_{t+1} L - 2)(\lambda_{t+1} - B_{t+1} \overline{K}) \\
& - B_{t+1}(\lambda_t L - 2)(\lambda_t - B_t \overline{K}) \\
< \; & \delta \lambda_t (1 - \lambda_t \overline{K})[\delta \lambda_t^2 L - \delta \lambda_t + \lambda_t L - 2] \\
& - B_{t+1}(\lambda_t L - 2)(\lambda_t - B_t \overline{K}) \\
< \; & 0 \\
if \ \delta \; \geq \; & \frac{L}{L+1} \qquad (32)
\end{aligned}
$$

∎

Next, we consider Case 3.

LEMMA 8. *If $\Delta_F \cap \Delta_D \neq \phi$, then the switching point can be given as a quadratic equation in*

$$(r_t, s_t) : \Phi_D(r_t, s_t) = \gamma_{D1} r_t^2 + \gamma_{D2} s_t^2 + \gamma_{D3} r_t s_t + \gamma_{D1} = 0$$

and

$$\Delta_F = \{(r_t, s_t) \in \mathbb{K}_{\Delta_{[0,1]}} \backslash \Delta_S^C | \Phi_D(r_t, s_t) < 0\}$$

Proof. Similar to the previous case, the expected value of RL along the equilibrium path can be represented as:

$$EV_t(r_{t-1}, s_{t-1}) = \mu_t p_t \left[R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right] + \delta(1-\mu_t)EV_{t+1}(r_t, s_t)$$

Now, with restricted funding, IC_t^{RL} binds on the equilibrium path, so:

$$\mu_t p_t \left[R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right] - K_t = \delta\left[\frac{\lambda_t p_t}{\lambda_{t+1}p_{t+1}} - (1-\mu_t)\right] EV_{t+1}(r_t, s_t) \tag{33}$$

Using the expression for $\lambda_{t+1}p_{t+1}$:

$$\begin{aligned} \lambda_{t+1}p_{t+1} &= \underbrace{\frac{\lambda_G - \lambda_U}{\lambda_G + \lambda_B - 2\lambda_U}}_{F}\lambda_t + \underbrace{\left(\lambda_U - 2\frac{\lambda_G - \lambda_U}{\lambda_G + \lambda_B - 2\lambda_U}\right)}_{G}p_t \\ &= F\lambda_t + Gp_t \end{aligned}$$

we can rewrite 33 as:

$$\begin{aligned} EV_{t+1}(r_t, s_t) &= \frac{\mu_t p_t\left[R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right] - K_t}{\delta\left[\frac{\lambda_t p_t}{F\lambda_t + Gp_t} - (1-\mu_t)\right]} \\ &= \frac{K_t\left[\lambda_t p_t R - \frac{\lambda_t p_t I}{p_t - vq_t} - \frac{p_t}{p_t - vq_t}\right]}{\delta\left[\frac{\lambda_t p_t}{\lambda_{t+1}p_{t+1}} - 1 + \lambda_t K_t\right]} \\ &= \frac{\left[\lambda_t p_t R - \frac{\lambda_t p_t I}{p_t - vq_t} - \frac{p_t}{p_t - vq_t}\right]}{\delta\left[\frac{\lambda_t p_t}{\lambda_{t+1}p_{t+1}K_t} - \frac{1}{K_t} + \lambda_t\right]}, \forall K_t \neq 0 \\ &= h_D(K_t) \end{aligned}$$

since $\lambda_t p_t > \lambda_{t+1}p_{t+1}$,

$$\frac{\partial h_D}{\partial K_t} > 0$$

Using the similar technique as in the previous case, we obtain the difference equation with restricted funding as:

$$g_D(K_t, K_{t+1}) = h_D(K_t) \tag{34}$$

where

$$\begin{aligned} EV_{t+1}(r_t, s_t) &= \mu_{t+1}p_{t+1}\left[R - \frac{1}{p_{t+1} - vq_{t+1}}\left(I + \frac{1}{\lambda_{t+1}}\right)\right] \\ &\quad +(1-\mu_{t+1})\frac{\mu_t p_t\left[R - \frac{1}{p_t - vq_t}\left(I + \frac{1}{\lambda_t}\right)\right] - K_t}{\left[\frac{\lambda_t p_t}{F\lambda_t + Gp_t} - (1-\mu_t)\right]} \\ &= g_D(K_t, K_{t+1}) \end{aligned} \tag{35}$$

with

$$\frac{\partial g_D}{\partial K_t} \leq 0, \frac{\partial g_D}{\partial K_{t+1}} > 0.$$

Then, by Implicit function theorem,

$$\begin{aligned} \frac{dK_{t+1}}{dK_t} &= -\frac{\frac{\partial g_D}{\partial K_t} - \frac{\partial h_D}{\partial K_t}}{\frac{\partial g_D}{\partial K_{t+1}}} \\ &\geq 0 \end{aligned}$$

Thus, the difference equation 34 expresses K_{t+1} as an increasing function of K_t. The fixed point can be written as the quadratic equation:

$$\Phi_D(r_t, s_t) = \gamma_{D1} r_t^2 + \gamma_{D2} s_t^2 + \gamma_{D3} r_t s_t + \gamma_{D1} = 0 \tag{36}$$

This denotes the switching point. Also, denote the area below the switching point as:

$$\Delta_F := \{(r_t, s_t) \in \mathbb{K}_{\Delta_{[0,1]}} \backslash \Delta_S^C | \Phi_D(r_t, s_t) \leq 0\}$$

If the condition is not satisfied, then at the switching point, the project is developed after obtaining patent. ∎

Next lemma shows that $\Delta_D \backslash \Delta_F$ shrinks as v increases, i.e. as CF becomes more ambiguity averse, the project receives full funding for longer horizon under Case 3, where at the switching point the project is developed if granted a patent.

Lemma 9. *If $\Delta_F \cap \Delta_D \neq \phi$, then $\Delta_D \backslash \Delta_F$ shrinks as v increases.*

Proof. The switching point 36 is given as:

$$\Phi_D(r_t, s_t) = g_D(\overline{K}, \overline{K}) - h_D(\overline{K}) = 0$$

$$\begin{aligned} \frac{\partial g_D}{\partial v} &= -\frac{K_{t+1} p_{t+1} q_{t+1} (I\lambda_{t+1} + 1)}{(p_{t+1} - v q_{t+1})^2} - (1 - \lambda_{t+1} K_{t+1}) \frac{K_t p_t q_t (I\lambda_t + 1)}{(p_t - v q_t)^2} \\ &< 0 \\ \frac{\partial h_D}{\partial v} &= -\frac{K_t p_t q_t (I\lambda_t + 1)}{(p_t - v q_t)^2} < 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial \Phi_D}{\partial v} &= \frac{\partial g_D}{\partial v} - \frac{\partial h_D}{\partial v} \\ &= -\frac{K_{t+1} p_{t+1} q_{t+1} (I\lambda_{t+1} + 1)}{(p_{t+1} - v q_{t+1})^2} \\ &\quad + \lambda_{t+1} K_{t+1} \frac{K_t p_t q_t (I\lambda_t + 1)}{(p_t - v q_t)^2} \\ &> 0 \end{aligned}$$

Also, using IFT,

$$\frac{\partial K_t}{\partial v} = -\frac{\frac{\partial \Phi_D}{\partial v}}{\frac{\partial \Phi_D}{\partial K_t}} \geq 0$$

Thus, $\Phi_D(r_t, s_t) = 0$ shifts, so, as v increases, the project receives full funding for a longer time if $\delta < \frac{L}{L+1}$, also funding level is weakly increasing in v after full funding stops.

Also,

$$\begin{aligned} \frac{\partial \Phi_D}{\partial (p_t - vq_t)} &= \frac{\partial g_D}{\partial (p_t - vq_t)} - \frac{\partial h_D}{\partial (p_t - vq_t)} \\ &< 0 \end{aligned}$$

So,

$$\frac{\partial K_t}{\partial (p_t - vq_t)} = -\frac{\frac{\partial \Phi_D}{\partial (p_t - vq_t)}}{\frac{\partial \Phi_D}{\partial K_t}} < 0$$

Intuitively, as v increases, the dynamic moral hazard decreases in the region where the project will be developed if patented. Thus, in the region $\Delta_D \backslash \Delta_F$, the project always receives full funding, and in the region Δ_F, investment gradually declines. This completes the proof of the proposition 3. ∎

Proof of Proposition 4. We will use the following two lemmata to derive the Policymaker's optima. ∎

LEMMA 10. *There exists a unique solution to the RAN optimization problem.*

Proof. The Policymaker's problem is recursively written as:

$$\begin{aligned} V^{RAN}(r,s) &= \max_{\Delta_H, \Delta_S, K^{RAN}} (P_t((r',s') \in \Delta_H)(pR - I) + \Pr((r',s') \in \Delta_S)L - K) \\ &\quad + \delta[\Pr((r^{'},s^{'}) \in \mathbb{K}_{\Delta[0,1]} \backslash (\Delta_H \cup \Delta_S)] EV^{RAN}(r',s') \end{aligned}$$

We can define the operator $T : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned} T(V^{RAN}) &= \max_{\Delta_H, \Delta_S, K^{RAN}} (\Pr((r',s') \in \Delta_H)(pR - I) + \Pr((r',s') \in \Delta_S)L - K) \\ &\quad + \delta[\Pr((r^{'},s^{'}) \in \mathbb{K}_{\Delta[0,1]} \backslash (\Delta_H \cup \Delta_S)] EV^{RAN}(r',s') \end{aligned}$$

Properties of T :

Monotonicity:

As $V(r,s) \leq V^1(r,s)$, $T(V) \leq T(V^1)$ $\forall (r,s) \in \mathbb{K}_{\Delta_{[0,1]}}$.

Discounting:

The discount factor $\delta \in (0,1)$ ensures that

$$[T(V + a)](r,s) \leq T(V)(r,s) + \delta a$$

for all $V, a \geq 0, (r,s) \in \mathbb{K}_{\Delta_{[0,1]}}$.

Hence T satisfies Blackwell's sufficiency conditions (Theorem 3.3 in Stokey (1989)), so it is a contraction. Then, by directly using the Contraction Mapping Theorem (Theorem 3.2 in Stokey (1989)), we show that T has exactly one fixed point V^{RAN} that solves the Policymaker's problem. ∎

Now, let us examine the optimal stopping rule. After observing the signal, based on the updated posterior $[r_t, s_t]$, the expected payoff is:

$$\max\{p_t R - I, L, \delta E_t V_{t+1}^{RAN}(r_t, s_t)\}$$

In order to solve for the RAN optima, let us define:

$$\begin{aligned} F_j(r_t, s_t) &= \text{ based on } [r_t, s_t], \text{ the maximum expected value if experimentation stops at } j \geq t \\ &= E_t\left[\delta^{j-t}\max\{p_j R - I, L\} - \sum_{s=t}^{j-1}\delta^{s-t}K_s\right] \end{aligned} \quad (37)$$

Define A_t as the set of posterior beliefs $[r_t, s_t]$ such that stopping at period t is weakly better than stopping at period $t+1$:

$$A_t = \{(r_t, s_t) | F_t \geq F_{t+1}\} \quad t = 1, 2, ..$$

we show that A_t s form a monotone sequence.

LEMMA 11. *If $F_t(r_t, s_t) \geq F_{t+1}(r_t, s_t)$, then $F_{t+1}(r_t, s_t) \geq F_{t+2}(r_t, s_t)$, i.e., $A_1 \subset A_2 \subset .. \cup_1^{\infty} A_n$, hence the region where stopping immediately is optimal forms a monotone sequence.*

Proof. Suppose $F_t(r_t, s_t) \geq F_{t+1}(r_t, s_t)$.

If the posterior belief at t is such that

$$(r_t, s_t) \in \Delta_H$$

, *i.e.*

$$p_t R - I > L$$

then

$$F_t(r_t, s_t) = p_t R - I$$

If (r_t, s_t) is such that at t^{th} period, after observing either $S_t = s_H$ or $S_t = s_L$, $p_{t+1}R - I > L$, then for all j,

$$F_{t+j}(r_t, s_t) = \delta^j\left[p_t R - I - \sum_{s=t+1}^{t+j} K_s\right] \leq F_t$$

so the result follows.

If (r_t, s_t) is such that at t^{th} period, after observing $S_t = s_H, p_{t+1|H}R - I > L$ and after observing $S_t = s_L, p_{t+1|L}R - I < L$,

$$\begin{aligned} F_{t+1}(r_t, s_t) &= \delta[\mu_{t+1}(p_{t+1}R - I) + (1-\mu_{t+1})L - K_{t+1}] \\ &\leq p_t R - I \\ &\Longleftrightarrow (1-\delta)\mu_{t+1}(p_{t+1|H}R - I) \\ &\geq \delta(1-\mu_{t+1})[L + p_{t+1|L}R - I] - K_{t+1}] \end{aligned} \tag{38}$$

then,

$$F_{t+2}(r_t, s_t) = E_t\left[\delta^2 \max\{p_{t+2}R - I, L\} - \delta^2 K_{t+2} - \delta K_{t+1}\right]$$

Thus,

$$\begin{aligned} & F_{t+2}(r_t, s_t) - F_{t+1}(r_t, s_t) \\ = \; & E_t\left[\delta^2 \max\{p_{t+2}R - I, L\} - \delta^2 K_{t+2}\right] - \delta[\mu_{t+1}(p_{t+1}R - I) + (1-\mu_{t+1})L] \\ \leq \; & \delta \left[\begin{array}{c} \delta\mu_{t+2}\mu_{t+1}(p_{t+2}|_{HH}R - I) + 2\delta(1-\mu_{t+1})\mu_{t+2}(p_{t+2}|_{LH}R - I) \\ +\delta(1-\mu_{t+1})(1-\mu_{t+2})L \\ -[\mu_{t+1}(p_{t+1}R - I) + (1-\mu_{t+1})L] \end{array} \right] - \delta^2 K_{t+2} \end{aligned}$$

$$\begin{aligned} = \; & \delta \left[\begin{array}{c} \delta\mu_{t+2}\mu_{t+1}(p_{t+2}|_{HH}R - I) + 2\delta(1-\mu_{t+1})\mu_{t+2}(p_{t+2}|_{LH}R - I) \\ +\delta(1-\mu_{t+1})(1-\mu_{t+2})L \\ -\mu_{t+2}\mu_{t+1}(p_{t+2}|_{HH}R - I) - (1-\mu_{t+2})\mu_{t+1}(p_{t+2}|_{LH}R - I) \\ -(1-\mu_{t+1})L \end{array} \right] - \delta^2 K_{t+2} \\ = \; & \delta \left[\begin{array}{c} (1-\mu_{t+1})\mu_{t+2}(p_{t+2}|_{LH}R - I)(2\delta - 1) \\ -(1-\delta)\mu_{t+2}\mu_{t+1}(p_{t+2}|_{HH}R - I) \\ -(1-\mu_{t+1})(1-\mu_{t+2})(1-\delta)L \end{array} \right] - \delta^2 K_{t+2} \\ = \; & \delta \left[\begin{array}{c} (1-\mu_{t+1})\mu_{t+2}(p_{t+2}|_{LH}R - I)\delta \\ -(1-\delta)\mu_{t+1}(p_{t+1}|_{H}R - I) \\ -(1-\mu_{t+1})(1-\mu_{t+2})(1-\delta)L \end{array} \right] - \delta^2 K_{t+2} \end{aligned}$$

$$\begin{aligned} \leq \; & \delta \left[\begin{array}{c} -\delta(1-\mu_{t+1})((p_{t+1}|_{H} - p_{t+2}|_{LH})R - I) \\ -L(1-\delta)(1-\mu_{t+1})(1-\mu_{t+2}) \\ -K_{t+1} \end{array} \right] - \delta^2 K_{t+2} \quad \text{(using 38)} \\ \leq \; & 0 \end{aligned}$$

Similarly, we can prove for the case when $(r_t, s_t) \in \Delta_L$. ∎

Proof of Proposition 4. By Lemma 11, $A_t s$ form a monotone sequence, the "One-stop ahead" rule is optimal, i.e., if stopping the experimentation process today is better than continuing experimenting for exactly one more period, then it is always optimal to stop today (Chow, Robbins, Siegmund (1971)). Using that, we obtain the optimal stopping rule, given in *Proposition 1.* The optimal stopping rules are found by equating F_t and F_{t+1}.

If $p_t R - I \geq L$,

$$F_t(r_t, s_t) = F_{t+1}(r_t, s_t)$$

yields the equation:

$$\beta_{H1} r_t + \beta_{H2} s_t = \beta_{H3} \tag{39}$$

and if $p_t R - I < L$, we obtain:

$$\beta_{S1} r_t + \beta_{S2} s_t = \beta_{S3} \tag{40}$$

where:

$$\begin{aligned}
\beta_{H1} &= R[1 - \delta(2\lambda_G - \lambda_U)] + \delta 2\overline{K}(I + L)(\lambda_G - \lambda_U) \\
\beta_{H2} &= R[1 - \delta\lambda_U] + \delta 2\overline{K}(I + L)(\lambda_U - \lambda_B) \\
\beta_{H3} &= 2I + 2\overline{K}\delta(1 - \lambda_B(I + L)) \\
\beta_{S1} &= \delta[R(2\lambda_G - \lambda_U) - 2\overline{K}(I + L)(\lambda_G - \lambda_U)] \\
\beta_{S2} &= \delta[R\lambda_U - 2\overline{K}(I + L)(\lambda_U - \lambda_B)] \\
\beta_{S3} &= 2L(1 - \delta) + 2\overline{K}\delta\lambda_B(I + L)
\end{aligned}$$

Under the parametric assumption 2, if $2\lambda_G - \lambda_U < \frac{1}{\delta}$, the two equations 39 and 40 yield downward sloping straight lines. ▮

Proof of Proposition 5. Since $\Delta_L \neq \phi$, the project is liquidated even after being patented in that region.

The optimal stopping region for the Policymaker is:

$$\Delta_S = \{(r_t, s_t) | \beta_{S1} r_t + \beta_{S2} s_t < \beta_{S3}\}$$

where:

$$\begin{aligned}
\beta_{S1} &= \delta[R(2\lambda_G - \lambda_U) - 2\overline{K}(I + L)(\lambda_G - \lambda_U)] \\
\beta_{S2} &= \delta[R\lambda_U - 2\overline{K}(I + L)(\lambda_U - \lambda_B)] \\
\beta_{S3} &= 2L(1 - \delta) + 2\overline{K}\delta\lambda_B(I + L)
\end{aligned}$$

For the partnership, the analogous region is:

$$\Delta_S^C = \{(r_t, s_t) | \lambda_t < \frac{2}{L}\}$$

At $r_t = s_t$, we can see the point on $\beta_{S1} r_t + \beta_{S2} s_t = \beta_{S3}$ is $r_S = s_S = \frac{L(1-\delta) + \overline{K}(I+L)\delta\lambda_B}{\delta R\lambda_G - \delta\overline{K}(I+L)(\lambda_G - \lambda_B)}$

and the point on $\lambda_t = \frac{2}{L}$ is $r_S^C = s_S^C = \frac{\frac{2}{L}-\lambda_B}{\lambda_G-\lambda_B}$. Even for $\delta = 1$, since $R > I$, it is always the case that (r_S^C, s_S^C) lies to the right of (r_S, s_S). Thus, $\Delta_S \subset \Delta_S^C$. ∎

APPENDIX B: AMBIGUITY FRAMEWORK

Denote the space of consequences as $\mathcal{X}$, which is a separable metric space with a topology that can be given by a metric making it complete. Let $C_b(\mathcal{X})$ denote the set of bounded, continuous functions on $\mathcal{X}$ with the supnorm topology, and $\Delta(\mathcal{X})$ be a weak* closed and separable, convex subset of the dual space of $C_b(\mathcal{X})$. Let $\mathbb{K}_{\Delta(\mathcal{X})}$ be the set of non-empty, compact, convex subsets of $\Delta(\mathcal{X})$ with the Hausdorff metric.

Then, a weak* continuous rational preference relation on $\mathbb{K}_{\Delta(\mathcal{X})}$ is a complete, transitive relation, $\succeq$, such that for all $B \in \mathbb{K}_{\Delta(\mathcal{X})}$, the sets $\{A : A \succ B\}$ and $\{B : B \succ A\}$ are open. The continuous linear preferences satisfy the Independence axiom given below.

AXIOM 1. *(Independence) For all* $A, B, C \in \boldsymbol{K}_{\Delta(\mathcal{X})}$, *and all* $\beta \in (0,1)$, $A \succeq B$ *if and only if* $\beta A + (1-\beta)C \succeq \beta B + (1-\beta)C$.

Then, the representation theorem shows that a continuous rational preference relation on $\mathbb{K}_{\Delta(\mathcal{X})}$ satisfies Axiom 1 if and only if it can be represented by a continuous linear functional.

THEOREM 1 (**Representation Theorem: Dumav and Stinchcombe, 2013**). *A continuous rational preference relation on* $\mathbb{K}_{\Delta(\mathcal{X})}$ *satisfies Axiom 1 if and only if it can be represented by a continuous linear functional* $L : \mathbb{K}_{\Delta(\mathcal{X})} \to \mathbb{R}$.

Using this representation theorem, we can define the value of ambiguous information analogous to the risky case.

In a risky case, for an expected utility maximizing decision maker, the information they will have when making a decision can be encoded in a posterior distribution, $\beta \in \Delta(\mathcal{X})$. The value of β is

$$V_u(\beta) = \max_{a \in A} \int u(a,x) d\beta(x), \quad where \;\; u : A \times X \to \mathbb{R}.$$

In risky case, a prior is a point $p \in \Delta(\mathcal{X})$, and an information structure is a dilation of p, that is, a distribution, $Q \in \Delta(\Delta(X))$, such that

$$\int \beta dQ(\beta) = p.$$

The value of the information structure is given by

$$V_u(Q) := \int_{\Delta(\mathcal{X})} V_u(\beta) dQ(\beta)$$

An information structure Q dominates Q' if for all u, $V_u(Q) \geq V_u(Q')$.

Analogously, for vNM utility maximizing decision maker facing an ambiguous problem, the information they will have when making a decision can be encoded in a set of posterior distributions, $B \in \mathbb{K}_{\Delta(\mathcal{X})}$.

The value of B is

$$V_U(B) = \max_{a \in A} U(\delta_a \times B)$$

where $U : A \times \mathbb{K}_{\Delta(\mathcal{X})} \rightarrow \mathbb{R}$ is a continuous linear functional on compact convex subsets of $\Delta(A \times \mathcal{X})$ of the form $\delta_a \times B$ (where δ_a is point mass on a).

A set-valued prior is a set $A \in \mathbb{K}_{\Delta(\mathcal{X})}$,and an information structure is a distribution, $Q \in \Delta(\mathbb{K}_{\Delta(\mathcal{X})})$, such that

$$\int_{\mathbb{K}_{\Delta(\mathcal{X})}} BdQ(B) = A.$$

Then, the value of the information structure Q is given by

$$V_U(Q) := \int_{\mathbb{K}_{\Delta(\mathcal{X})}} V_U(B)dQ(B).$$

As above, an information structure Q dominates Q' if for all U, $V_U(Q) \geq V_U(Q')$.

This framework follows the standard Bayesian approach and models information structures as dilations. By contrast, previous work has limited the class of priors, A, and then studied a special class of dilations of each $p \in A$. The set of A for which this can be done is non-generic in both the measure theoretic and the topological sense, and the problems that one can consider are limited to ones in which the decision maker will learn only that the true value belong to some $E \subset \mathcal{X}$.

In this approach, A is expressed as a convex combination of/integral of B's in $\mathbb{K}_{\Delta(\mathcal{X})}$, and this is what makes the problem tractable and brings about dynamic consistency.

In a two-consequence case which will be considered in this paper, this approach simplifies to representing preferences as linear functionals in a simplex. If $\mathcal{X} = \{$*Good*, *Bad*$\}$, then $\mathbb{K}_{\Delta(\mathcal{X})}$ is the class of non-empty closed, convex subsets of the probabilities represented as a simplex:

$$\mathbb{K}_{\Delta(\mathcal{X})} = \{[p - r, p + r] : 0 \leq p - r \leq p + r \leq 1\}.$$

In this case, continuous linear functionals on the convex sets of probabilities must be of the form

$$U([a, b]) = u_1 a + u_2 b$$

for $u_1,u_2 \in \mathbb{R}$.

Rewriting $[a, b]$ as $[p - r, p + r]$, where $p = \frac{a+b}{2}$ and $q = \frac{b-a}{2}$ yields

$$U([p - r, p + r]) = (u_1 + u_2)p - (u_1 - u_2)r = p - vr$$

with $v = u_1 - u_2$ measuring the trade-off between risk and ambiguity, $v > 0$ represents ambiguity averse attitude.

Graphically, a set-valued prior $[a, b]$ can be represented as a point in the simplex T with three vertices, $(0, 0)$ representing *Bad* state, $(1, 1)$ representing *Good* state and the new epistemic state "*Unknowable*" represented by the vertex $(0, 1)$. Each $[a, b]$ has a unique representation as

$$(a, b) = w_{1,1}(1, 1) + w_{0,1}(0, 1) + (1 - w_{1,1} - w_{0,1})(0, 0)$$

solving,

$$w_{1,1} = a, w_{0,1} = b - a, w_{0,0} = 1 - b.$$

Thus, the prior $[a, b]$ assigns weight a on $(1, 1)$, $1 - b$ on $(0, 0)$ and $b - a$ on the state $(0, 1)$, i. e. , according to the decision maker, the evidence is thoroughly inconclusive with probability $(b - a)$.

In this setting, a signal is a dilation of the prior which enables Bayesian updating of the weights on each vertex of T. For example, if a binary signal $s \in \{s_1, s_2\}$, $\Pr(s = s_1|Good) = \eta_{1,1}$; $\Pr(s = s_1|Bad) = \eta_{0,0}$ and $\Pr(s = s_1|Unknowable) = \eta_{0,1}$, then the decision maker with prior $[a, b]$ updates his prior after observing s_1 as follows:

$$\begin{aligned}
\Pr(Good|s_1) &= \frac{\eta_{1,1}a}{\eta_{1,1}a + \eta_{0,0}(1 - b) + \eta_{0,1}(b - a)} \\
\Pr(Bad|s_1) &= \frac{\eta_{0,0}(1 - b)}{\eta_{1,1}a + \eta_{0,0}(1 - b) + \eta_{0,1}(b - a)} \\
\Pr(Unknowable|s_1) &= \frac{\eta_{0,1}(b - a)}{\eta_{1,1}a + \eta_{0,0}(1 - b) + \eta_{0,1}(b - a)}
\end{aligned}$$

Hence, posterior

$$[a', b']|_{s=s_1} = \left[\frac{\eta_{1,1}a}{\eta_{1,1}a + \eta_{0,0}(1 - b) + \eta_{0,1}(b - a)}, 1 - \frac{\eta_{0,0}(1 - b)}{\eta_{1,1}a + \eta_{0,0}(1 - b) + \eta_{0,1}(b - a)}\right]$$

In this paper we use this framework to model ambiguous decision making in the innovation process.