# Extensions of the Schur majorisation inequalities 

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#### Abstract

Let $\lambda_{j}$ and $a_{j j}, 1 \leq j \leq n$, be the eigenvalues and the diagonal entries of a Hermitian matrix $A$, both enumerated in the increasing order. We prove some inequalities that are stronger than the Schur majorisation inequalities $\sum_{j=1}^{r} \lambda_{j} \leq \sum_{j=1}^{r} a_{j j}, 1 \leq r \leq n$.


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## 1. Introduction

Let $A$ be an $n \times n$ complex Hermitian matrix. Let the eigenvalues and the diagonal entries of $A$ both be enumerated in increasing order as

$$
\begin{equation*}
\lambda_{1}(A) \leq \lambda_{2}(A) \leq \cdots \leq \lambda_{n}(A), \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{11} \leq a_{22} \leq \cdots \leq a_{n n}, \tag{1.2}
\end{equation*}
$$

respectively. We then have

$$
\begin{equation*}
\lambda_{1}(A) \leq a_{11} \quad \text { and } \quad \lambda_{n}(A) \geq a_{n n} . \tag{1.3}
\end{equation*}
$$

These inequalities are included in the Schur majorisation inequalities that say: for every $1 \leq r \leq n$

$$
\begin{equation*}
\sum_{j=1}^{r} \lambda_{j}(A) \leq \sum_{j=1}^{r} a_{j j}, \tag{1.4}
\end{equation*}
$$

with equality in the case $r=n$. These inequalities are of fundamental importance in matrix analysis and have been the subject of intensive work. See, e.g. Bhatia [1], Horn and Johnson [4] and Marshal and Olkin [5].

In this note we obtain some inequalities that are stronger than (1.3) and (1.4). These give estimates of eigenvalues in terms of quantities easily computable from the entries of $A$.

Given the $n \times n$ Hermitian matrix $A=\left[a_{i j}\right]$, let

$$
\begin{equation*}
r_{i}=\sum_{j \neq i}\left|a_{i j}\right|, \quad 1 \leq i \leq n \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
q_{i}=\sum_{j \neq i}\left|a_{i j}\right|^{2}, \quad 1 \leq i \leq n . \tag{1.6}
\end{equation*}
$$

A permutation similarity does not change either the eigenvalues or the diagonal entries of $A$. Nor does it change the quantities $r_{i}$ and $q_{i}$. We assume that such a permutation similarity has been performed and the ordering (1.2) for diagonal entries has been achieved. To rule out trivial cases, we assume that $A$ is not a diagonal matrix.

Our first theorem is a strengthening of the inequalities (1.3).

Theorem 1 For every $n \times n$ Hermitian matrix $A$, we have

$$
\begin{align*}
& \lambda_{1}(A) \leq a_{11}-\frac{q_{1}}{\max _{i}\left(a_{i i}+r_{i}\right)-a_{11}},  \tag{1.7}\\
& \lambda_{n}(A) \geq a_{n n}+\frac{q_{n}}{a_{n n}-\min _{i}\left(a_{i i}-r_{i}\right)} . \tag{1.8}
\end{align*}
$$

The next two theorems give inequalities stronger than (1.4).

Theorem 2 Let $A$ be an $n \times n$ Hermitian matrix. Then for $1 \leq r \leq n-1$ and $r<t \leq n$, we have

$$
\begin{equation*}
\sum_{i=1}^{r} \lambda_{i}(A) \leq \sum_{i=1}^{r} a_{i i}-\frac{\sum_{s=1}^{r}\left|a_{t s}\right|^{2}}{a_{t t}-\min _{i=1, \ldots, r, t}\left(a_{i i}-\sum_{\substack{s=1 \\ s \neq i}}^{r+1}\left|a_{i s}\right|\right)} \tag{1.9}
\end{equation*}
$$

Theorem 3 Let $A$ be an $n \times n$ Hermitian matrix. Then for $1 \leq r \leq n-1,1 \leq k \leq r$, and $r<t \leq n$, we have

$$
\begin{equation*}
\sum_{i=1}^{r} \lambda_{i}(A) \leq \sum_{i=1}^{r} a_{i i}-\frac{\sqrt{\left(a_{t t}-a_{k k}\right)^{2}+4\left|a_{t k}\right|^{2}}-\left(a_{t t}-a_{k k}\right)}{2} . \tag{1.10}
\end{equation*}
$$

## 2. Proofs

Our proofs rely upon two basic theorems of matrix analysis. Let $\mathbb{M}(n)$ be the algebra of all $n \times n$ complex matrices and let $\Phi: \mathbb{M}(n) \rightarrow \mathbb{M}(k)$ be a positive unital linear map, [3]. Then the Bhatia-Davis inequality [2] says that for every Hermitian matrix $A$ whose spectrum is contained in the interval $[m, M]$, we have

$$
\begin{equation*}
\Phi\left(A^{2}\right)-\Phi(A)^{2} \leq(M I-\Phi(A))(\Phi(A)-m I) \leq\left(\frac{M-m}{2}\right)^{2} I . \tag{2.1}
\end{equation*}
$$

Cauchy's interlacing principle says that if $A_{r}$ is an $r \times r$ principal submatrix of $A$, then

$$
\begin{equation*}
\lambda_{j}(A) \leq \lambda_{j}\left(A_{r}\right), \quad 1 \leq j \leq r \tag{2.2}
\end{equation*}
$$

See Chapter III of [1] for this and other facts used here.

### 2.1. Proof of Theorem 1

Let $\varphi: \mathbb{M}(n) \rightarrow \mathbb{C}$ be a positive unital linear functional and let the eigenvalues of Hermitian element $A \in \mathbb{M}(n)$ be arranged as in (1.1). From the first inequality (2.1), we have

$$
\begin{equation*}
\varphi\left(A^{2}\right)-\varphi(A)^{2} \leq\left(\lambda_{n}(A)-\varphi(A)\right)\left(\varphi(A)-\lambda_{1}(A)\right) \tag{2.3}
\end{equation*}
$$

Suppose $\lambda_{n}(A) \neq \varphi(A)$. Then, from (2.3), we have

$$
\begin{equation*}
\lambda_{1}(A) \leq \varphi(A)-\frac{\varphi\left(A^{2}\right)-\varphi(A)^{2}}{\lambda_{n}(A)-\varphi(A)} \tag{2.4}
\end{equation*}
$$

Further, by the Gersgorin disk theorem, we have

$$
\begin{equation*}
\lambda_{n}(A) \leq \max _{i}\left(a_{i i}+r_{i}\right) . \tag{2.5}
\end{equation*}
$$

Combining (2.4) and (2.5), we get

$$
\begin{equation*}
\lambda_{1}(A) \leq \varphi(A)-\frac{\varphi\left(A^{2}\right)-\varphi(A)^{2}}{\max _{i}\left(a_{i i}+r_{i}\right)-\varphi(A)} \tag{2.6}
\end{equation*}
$$

Choose $\varphi(A)=a_{11}$. Then, $\varphi$ is a positive unital linear functional and $\varphi\left(A^{2}\right)-\varphi(A)^{2}=$ $q_{1}$. So, (2.6) yields (1.7).
Suppose $\lambda_{n}(A)=\varphi(A)=a_{11}$. Then, from (1.2) and (1.3), we have $a_{11}=a_{22}=\cdots=a_{n n}$ and from (2.3), $\varphi\left(A^{2}\right)-\varphi(A)^{2}=0$. Therefore, $q_{i}=0$ for all $i=1,2, \ldots, n$. But then $A$ is a scalar matrix.

The inequality (1.8) follows on using similar arguments. The derivation requires lower bound of $\lambda_{n}(A)$ from (2.3) which is analogous to (2.4), $\lambda_{1}(A) \geq \min _{i}\left(a_{i i}-r_{i}\right)$ and $\varphi(A)=a_{n n}$.

### 2.2. Proof of Theorem 2

The trace of $A$ is the sum of the eigenvalues of $A$. Therefore,

$$
\begin{equation*}
\lambda_{n}(A)=\operatorname{tr} A-\sum_{i=1}^{n-1} \lambda_{i}(A) . \tag{2.7}
\end{equation*}
$$

Combining (1.8) and (2.7), we find that

$$
\begin{equation*}
\sum_{i=1}^{n-1} \lambda_{i}(A) \leq \sum_{i=1}^{n-1} a_{i i}-\frac{q_{n}}{a_{n n}-\min _{i}\left(a_{i i}-r\right)} . \tag{2.8}
\end{equation*}
$$

Apply (2.8) to the principal submatrix $P$ of $A$ containing diagonal entries $a_{11}, a_{22}, \ldots, a_{r r}, a_{t t}$, we get that

$$
\begin{equation*}
\sum_{i=1}^{r} \lambda_{i}(P) \leq \sum_{i=1}^{r} a_{i i}-\frac{\sum_{s=1}^{r}\left|a_{t s}\right|^{2}}{a_{t t}-\min _{i=1, \ldots, r, t}\left(a_{i i}-\sum_{\substack{s=1 \\ s \neq i}}^{r+1}\left|a_{i s}\right|\right)} \tag{2.9}
\end{equation*}
$$

By the interlacing inequalities (2.2), $\sum_{i=1}^{r} \lambda_{i}(A) \leq \sum_{i=1}^{r} \lambda_{i}(P)$. So, (2.9) gives (1.9).

### 2.3. Proof of Theorem 3

By the Cauchy interlacing principle $\sqrt{2.2}$, the largest eigenvalue of $A$ is greater than or equal to the largest eigenvalue of any $2 \times 2$ principal submatrix of $A$. Further, the eigenvalues of $\left[\begin{array}{ll}a_{r r} & a_{r s} \\ \overline{a_{r s}} & a_{s s}\end{array}\right]$ are $\frac{1}{2}\left(a_{r r}+a_{s s} \pm \sqrt{\left(a_{r r}-a_{s s}\right)^{2}+4\left|a_{r s}\right|^{2}}\right)$. On using these two facts, we see that

$$
\begin{equation*}
\lambda_{n}(A) \geq a_{n n}+\frac{\sqrt{\left(a_{n n}-a_{k k}\right)^{2}+4\left|a_{k n}\right|^{2}}-\left(a_{n n}-a_{k k}\right)}{2} \tag{2.10}
\end{equation*}
$$

for all $k=1,2, \ldots, n-1$. Combining (2.7) and (2.10), we find that

$$
\begin{equation*}
\sum_{i=1}^{n-1} \lambda_{i}(A) \leq \sum_{i=1}^{n-1} a_{i i}-\frac{\sqrt{\left(a_{n n}-a_{k k}\right)^{2}+4\left|a_{k n}\right|^{2}}-\left(a_{n n}-a_{k k}\right)}{2} . \tag{2.11}
\end{equation*}
$$

Apply (2.11) to the principal submatrix $Q$ of $A$ containing $a_{11}, a_{22}, \ldots, a_{r r}, a_{t t}$, we find that for $k=1,2, \ldots, r$, we have

$$
\begin{equation*}
\sum_{i=1}^{r} \lambda_{i}(Q) \leq \sum_{i=1}^{r} a_{i i}-\frac{\sqrt{\left(a_{t t}-a_{k k}\right)^{2}+4\left|a_{t k}\right|^{2}}-\left(a_{t t}-a_{k k}\right)}{2} \tag{2.12}
\end{equation*}
$$

The inequality (2.12) yields (1.10), on using the interlacing inequalities (2.2).
We show by means of an example that (1.9) and (1.10) are independent.

Example 1 Let

$$
A=\left[\begin{array}{lll}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 3
\end{array}\right], \quad B=\left[\begin{array}{lll}
1 & 2 & 3 \\
2 & 1 & 4 \\
3 & 4 & 1
\end{array}\right]
$$

Then (1.9) gives the estimate $\lambda_{1}(A)+\lambda_{2}(A)<\frac{10}{3}$, while (1.10) gives the weaker estimate $\frac{9-\sqrt{5}}{2}$ for the same quantity. On the other hand from (1.9) we get that $\lambda_{1}(B)+\lambda_{2}(B)<$ $-\frac{11}{7}$, while from (1.10) we see that the same quantity is not bigger than -2 .

## References

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