Extensions of the Schur majorisation inequalities

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Abstract

Let λ_j and a_{jj} , $1 \le j \le n$, be the eigenvalues and the diagonal entries of a Hermitian matrix *A*, both enumerated in the increasing order. We prove some inequalities that are stronger than the Schur majorisation inequalities $\sum_{j=1}^{r} \lambda_j \le \sum_{j=1}^{r} a_{jj}$, $1 \le r \le n$.

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1. Introduction

Let *A* be an $n \times n$ complex Hermitian matrix. Let the eigenvalues and the diagonal entries of *A* both be enumerated in increasing order as

$$\lambda_1(A) \le \lambda_2(A) \le \dots \le \lambda_n(A), \tag{1.1}$$

and

$$a_{11} \le a_{22} \le \dots \le a_{nn},\tag{1.2}$$

respectively. We then have

$$\lambda_1(A) \le a_{11} \quad \text{and} \quad \lambda_n(A) \ge a_{nn}.$$
 (1.3)

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These inequalities are included in the Schur majorisation inequalities that say: for every $1 \le r \le n$

$$\sum_{j=1}^{r} \lambda_j(A) \le \sum_{j=1}^{r} a_{jj},\tag{1.4}$$

with equality in the case r = n. These inequalities are of fundamental importance in matrix analysis and have been the subject of intensive work. See, e.g. Bhatia [1], Horn and Johnson [4] and Marshal and Olkin [5].

In this note we obtain some inequalities that are stronger than (1.3) and (1.4). These give estimates of eigenvalues in terms of quantities easily computable from the entries of *A*.

Given the $n \times n$ Hermitian matrix $A = [a_{ij}]$, let

$$r_i = \sum_{j \neq i} \left| a_{ij} \right|, \quad 1 \le i \le n \tag{1.5}$$

and

$$q_i = \sum_{j \neq i} |a_{ij}|^2, \quad 1 \le i \le n.$$
 (1.6)

A permutation similarity does not change either the eigenvalues or the diagonal entries of *A*. Nor does it change the quantities r_i and q_i . We assume that such a permutation similarity has been performed and the ordering (1.2) for diagonal entries has been achieved. To rule out trivial cases, we assume that *A* is not a diagonal matrix.

Our first theorem is a strengthening of the inequalities (1.3).

Theorem 1 For every $n \times n$ Hermitian matrix A, we have

$$\lambda_1(A) \le a_{11} - \frac{q_1}{\max_i(a_{ii} + r_i) - a_{11}},\tag{1.7}$$

$$\lambda_n(A) \ge a_{nn} + \frac{q_n}{a_{nn} - \min_i(a_{ii} - r_i)}.$$
(1.8)

The next two theorems give inequalities stronger than (1.4).

Theorem 2 Let A be an $n \times n$ Hermitian matrix. Then for $1 \le r \le n-1$ and $r < t \le n$, we have

$$\sum_{i=1}^{r} \lambda_i(A) \le \sum_{i=1}^{r} a_{ii} - \frac{\sum_{s=1}^{r} |a_{ts}|^2}{a_{tt} - \min_{i=1,\dots,r,t} \left(a_{ii} - \sum_{\substack{s=1\\s \neq i}}^{r+1} |a_{is}| \right)}.$$
(1.9)

Theorem 3 Let A be an $n \times n$ Hermitian matrix. Then for $1 \le r \le n-1$, $1 \le k \le r$, and $r < t \le n$, we have

$$\sum_{i=1}^{r} \lambda_i(A) \le \sum_{i=1}^{r} a_{ii} - \frac{\sqrt{(a_{tt} - a_{kk})^2 + 4|a_{tk}|^2} - (a_{tt} - a_{kk})}{2}.$$
 (1.10)

2. Proofs

Our proofs rely upon two basic theorems of matrix analysis. Let $\mathbb{M}(n)$ be the algebra of all $n \times n$ complex matrices and let $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ be a positive unital linear map, [3]. Then the Bhatia-Davis inequality [2] says that for every Hermitian matrix A whose spectrum is contained in the interval [m, M], we have

$$\Phi(A^2) - \Phi(A)^2 \le (MI - \Phi(A)) \left(\Phi(A) - mI\right) \le \left(\frac{M - m}{2}\right)^2 I.$$
(2.1)

Cauchy's interlacing principle says that if A_r is an $r \times r$ principal submatrix of A, then

$$\lambda_j(A) \le \lambda_j(A_r), \quad 1 \le j \le r.$$
 (2.2)

See Chapter III of [1] for this and other facts used here.

2.1. Proof of Theorem 1

Let $\varphi : \mathbb{M}(n) \to \mathbb{C}$ be a positive unital linear functional and let the eigenvalues of Hermitian element $A \in \mathbb{M}(n)$ be arranged as in (1.1). From the first inequality (2.1), we have

$$\varphi(A^2) - \varphi(A)^2 \le (\lambda_n(A) - \varphi(A)) \left(\varphi(A) - \lambda_1(A)\right).$$
(2.3)

Suppose $\lambda_n(A) \neq \varphi(A)$. Then, from (2.3), we have

$$\lambda_1(A) \le \varphi(A) - \frac{\varphi(A^2) - \varphi(A)^2}{\lambda_n(A) - \varphi(A)}.$$
(2.4)

Further, by the Gersgorin disk theorem, we have

$$\lambda_n(A) \le \max_i (a_{ii} + r_i). \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\lambda_1(A) \le \varphi(A) - \frac{\varphi(A^2) - \varphi(A)^2}{\max_i(a_{ii} + r_i) - \varphi(A)}.$$
(2.6)

Choose $\varphi(A) = a_{11}$. Then, φ is a positive unital linear functional and $\varphi(A^2) - \varphi(A)^2 = q_1$. So, (2.6) yields (1.7).

Suppose $\lambda_n(A) = \varphi(A) = a_{11}$. Then, from (1.2) and (1.3), we have $a_{11} = a_{22} = \cdots = a_{nn}$ and from (2.3), $\varphi(A^2) - \varphi(A)^2 = 0$. Therefore, $q_i = 0$ for all i = 1, 2, ..., n. But then A is a scalar matrix.

The inequality (1.8) follows on using similar arguments. The derivation requires lower bound of $\lambda_n(A)$ from (2.3) which is analogous to (2.4), $\lambda_1(A) \ge \min_i (a_{ii} - r_i)$ and $\varphi(A) = a_{nn}$.

2.2. Proof of Theorem 2

The trace of A is the sum of the eigenvalues of A. Therefore,

$$\lambda_n(A) = \operatorname{tr} A - \sum_{i=1}^{n-1} \lambda_i(A).$$
(2.7)

Combining (1.8) and (2.7), we find that

$$\sum_{i=1}^{n-1} \lambda_i(A) \le \sum_{i=1}^{n-1} a_{ii} - \frac{q_n}{a_{nn} - \min_i(a_{ii} - r)}.$$
(2.8)

Apply (2.8) to the principal submatrix *P* of *A* containing diagonal entries $a_{11}, a_{22}, ..., a_{rr}, a_{tt}$, we get that

$$\sum_{i=1}^{r} \lambda_i(P) \le \sum_{i=1}^{r} a_{ii} - \frac{\sum_{s=1}^{r} |a_{ts}|^2}{a_{tt} - \min_{i=1,\dots,r,t} \left(a_{ii} - \sum_{\substack{s=1\\s \neq i}}^{r+1} |a_{is}| \right)}.$$
(2.9)

By the interlacing inequalities (2.2), $\sum_{i=1}^{r} \lambda_i(A) \leq \sum_{i=1}^{r} \lambda_i(P)$. So, (2.9) gives (1.9). \Box

2.3. Proof of Theorem 3

By the Cauchy interlacing principle (2.2), the largest eigenvalue of A is greater than or equal to the largest eigenvalue of any 2×2 principal submatrix of A. Further, the eigenvalues of $\begin{bmatrix} a_{rr} & a_{rs} \\ \overline{a_{rs}} & a_{ss} \end{bmatrix}$ are $\frac{1}{2} \left(a_{rr} + a_{ss} \pm \sqrt{(a_{rr} - a_{ss})^2 + 4|a_{rs}|^2} \right)$. On using these two facts, we see that

$$\lambda_n(A) \ge a_{nn} + \frac{\sqrt{(a_{nn} - a_{kk})^2 + 4|a_{kn}|^2} - (a_{nn} - a_{kk})}{2}$$
(2.10)

for all k = 1, 2, ..., n - 1. Combining (2.7) and (2.10), we find that

$$\sum_{i=1}^{n-1} \lambda_i(A) \le \sum_{i=1}^{n-1} a_{ii} - \frac{\sqrt{(a_{nn} - a_{kk})^2 + 4|a_{kn}|^2} - (a_{nn} - a_{kk})}{2}.$$
 (2.11)

Apply (2.11) to the principal submatrix Q of A containing $a_{11}, a_{22}, ..., a_{rr}, a_{tt}$, we find that for k = 1, 2, ..., r, we have

$$\sum_{i=1}^{r} \lambda_i(Q) \le \sum_{i=1}^{r} a_{ii} - \frac{\sqrt{(a_{tt} - a_{kk})^2 + 4|a_{tk}|^2} - (a_{tt} - a_{kk})}{2}.$$
 (2.12)

The inequality (2.12) yields (1.10), on using the interlacing inequalities (2.2). \Box

We show by means of an example that (1.9) and (1.10) are independent.

Example 1 Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}.$$

Then (1.9) gives the estimate $\lambda_1(A) + \lambda_2(A) < \frac{10}{3}$, while (1.10) gives the weaker estimate $\frac{9-\sqrt{5}}{2}$ for the same quantity. On the other hand from (1.9) we get that $\lambda_1(B) + \lambda_2(B) < -\frac{11}{7}$, while from (1.10) we see that the same quantity is not bigger than -2.

References

- [1] Bhatia R., Matrix Analysis, Springer, New York, (1997).
- [2] Bhatia R., Davis C., A better bound on the variance, Amer. Math. Monthly, 107, (2000), 353-357.
- [3] Bhatia R., Positive Definite Matrices, Princeton University Press, (2007).
- [4] Horn R.A., Johnson C.R., Matrix Analysis, Cambridge University Press, (2013).
- [5] Marshal A.W., Olkin I., *Inequalities: Theory of Majorisation and its applications*, Academic Press, (1979).